HW #6, due 5-9

Math 506A

1. Let q be a prime power. Using the Orbit Formula from the fall, work out the orders of the Grassmannian $\mathrm{Gr}_d n$ of subspaces of dimension d in $\boldsymbol{F}_q^n$, a vector space of dimension n over the field $\boldsymbol{F}_q$ with q elements (this Grassmannian is a finite set). Also work out the order of the flag variety of $\boldsymbol{F}_q^n$

2. Decompose the flag variety of $\boldsymbol{C}^n$ into suborbits for the Borel subgroup B of upper triangular matrices in $GL_n(\boldsymbol{C})$, showing that each suborbit is isomorphic to a suitable affine space $\boldsymbol{C}^m$ for some m.

3. It was shown in class that an ideal I of a commutative ring R is contained in the Jacobson radical of R (the intersection of its maximal ideals) if and only if $1+i$ is a unit in R for all $i \in I$. Use this to show that if R is complete with respect to an ideal I, then I is contained in every maximal ideal of R..

4. If I is a finitely generated ideal of a ring R then show that the ideal $IR[[x]]$ is the ideal of all power series in x over R having all coefficients in I. Find an example where I is not finitely generated and this conclusion fails.

5. Let M be a module over a ring R that is complete with respect to an ideal I. We say that M is *separated* if $\cap_k I^k M = 0$. If M is separated and the images of $m_1, \ldots, m_n \in M$ generate M/IM, then show that $m_1, \ldots, m_n$ generated M.

Look at Chapter 7 of Eisenbud's Commutative Algebra book.