HW #5, DUE 2-10

1. Let R be a Noetherian ring. Show that the polynomial ring $R[x]$ is Noetherian. (You must show that every nonzero ideal I of $R[x]$ is finitely generated; given I, let L consist of all leading coefficients of all nonzero elements of I. Show that L is an ideal of R; if it is generated by the leading terms $a_1, \ldots, a_n$ of the polynomials $p_1, \ldots, p_n \in I$, then show that I is the sum of $(p_1, \ldots, p_n)$ and the intersection J of I with the set of polynomials in $R[x]$ of degree at most N, where N is the maximum of the degrees of the p_i. Finally, show that J is an R-submodule of a free R-module of finite rank, so is finitely generated already as an R-module.). Deduce that the polynomial ring $K[x_1, \ldots, x_n]$ in any finite number of variables over a field K is Noetherian.

2. Similarly show that the power series ring $R[[x]]$ is Noetherian whenever R is; arguing similarly to the previous problem, except that L consists of the nonzero coefficients of the *lowest* power of x in any nonzero element of I.

3. Let p be a prime number and q a monic polynomial in $\mathbb{Z}_p[x]$ that has a root of multiplicity one in $\mathbb{Z}_p$. Show that q also has a root in the p-adic integers.

4. Let K be an algebraically closed field and V the subvariety of K^2 of zeros of the single polynomial $xy - 1$. Exhibit the coordinate ring $K[V]$ as an integral extension of a suitable polynomial ring over K and write down the corresponding morphism realizing V as a ramified finite cover of the affine line K^1. What are the sizes of the fibers of this morphism?

5. Again let K be algebraically closed and let W be the variety of zeros in K^2 of the single polynomial $x^2 - y^3$. Exhibit a morphism from the affine line K^1 onto W that is bijective, but whose inverse is not a morphism (since the associated algebra homomorphism from $K[W]$ to $K[K^1] = K[x]$ is not an isomorphism.)