FINAL EXAM–MATH 505A

1. Let L be a Galois extension of a field K of degree 4. What is the largest possible number of fields there could be strictly between K and L? What is the smallest possible number of such fields? Give examples showing that the bounds you claim are attained.

2. Let $\omega = e^{2\pi i/7}$ be a primitive 7th root of 1 in $\mathbb{C}$ and let K be the cyclotomic field $\mathbb{Q}[\omega]$. Find values of the exponents a, b, c such that $\alpha = \omega^a + \omega^b + \omega^c$ generates a quadratic extension L of $\mathbb{Q}$ inside K, by identifying the Galois group of K over $\mathbb{Q}$ explicitly and deciding to which subgroup of this group L should correspond.

3. By using a formula in class for the second cohomology group $H^2(\mathbb{Z}_n, A)$ of the cyclic group $\mathbb{Z}_n$ with coefficients in a module A, together with facts about finite fields, show that the only finite division algebras over the field $F = F_p$ with a prime number p of elements are finite extensions of F (Wedderburn's Theorem).

4. Let $R = \mathbb{Z}[\sqrt{-3}]$ be the subring of $\mathbb{C}$ generated by $\mathbb{Z}$ and $\sqrt{-3}$. Enlarge the the principal ideal (2) of R to a prime ideal P and show that (2) lies strictly between P^2 and P. Deduce that R is *not* a Dedekind domain.

5. Give a necessary and sufficient condition for a Dedekind domain R to admit a finitely generated nonfree projective module.

6. Give an example of a finite non-Galois extension L of a field K such that there are more fields between K and L than subgroups of the automorphism group G of L over K. Also give an example of an *infinite Galois* extension L' of a field K' such that there are more subgroups of the Galois group G' of L' over K' than fields between K' and L'.

7. By looking at quotients of polynomial rings in infinitely many variables over a field, give an example of a non-Noetherian, non-Artinian ring R such that every prime ideal of R is maximal.

8. Classify the finite-dimensional semisimple algebras over $\mathbb{C}$, the field of complex numbers.