Lecture 3-5: Discrete valuation rings

March 5, 2025

Previously I have considered a class of rings slightly more general than PIDs, namely Dedekind domains; now I round out the course by looking at a very special class of PIDs, namely discrete valuation rings. These are often called DVRs; but since they are all in fact integral domains, they could with equal justification be called DVDs (though that abbreviation has alas been taken).

Definition; see Theorem 7, part 2, p. 757

A *discrete valuation ring* (or domain) is a PID with only one nonzero prime ideal P.

Letting x be a generator of P, we know that every nonzero ideal is a product of prime ideals, from which it follows at once that the only nonzero ideals of R are the principal ones (x^n) generated by powers of x. Every nonzero element of R can be uniquely written as $x^n u$ for nonnegative integer n and unit u. In particular, R is a local ring with unique maximal ideal P. Taking the quotient $R/(x^n)$ by any power of P gives a local Artinian ring.

Such rings at first seem very special, which indeed they are; but they are not as rare as you might think. To show why this is so, I give a separate definition of valuation on a field.

Definition, p. 755

A (discrete) *valuation* on a field K is a map $v : K^* \to \mathbb{Z}$ such that $v(xy) = v(x) + v(y)$ for $x, y \in K^*$ and $v(x+y) \geq \min(v(x), v(y))$ if $x, y, x+y \in K^*$. It is sometimes convenient to extend v to all of K by decreeing that $v(0) = \infty$. The *valuation ring* corresponding to K and v is $R = \{x \in K^* : v(x) \geq 0\} \cup \{0\}$.

Any valuation v on K satisfies $v(1) = v(1 \cdot 1) = 0$, whence $v(u) = 0$ if and only if u is a unit in the valuation ring R (assuming, as we do, that v is not identically 0) and choose $x \in R$ with the smallest positive value $n = v(x)$. Then any nonzero $y \in R$ has $0 \leq v(yx^{-m}) < n$ for some m, whence $u = yx^{-m}$ lies in R and must be a unit.

It follows that R is a PID such that any nonzero ideal takes the form (x^n) for a unique n (Proposition 5, p. 756). Indeed, given a nonzero ideal I, choose $y \in I$ with $v(y) = n$ minimal. Then $y = x^n u$ for some unit u, so that $(y) = (x^n)$. Any $z \in I$ has $v(z) \geq n$ by the choice of y, so $v(zy^{-1}) \geq 0$ and $zy^{-1} \in R, z \in (y)$, and $I = (y)$, as claimed. Thus R is a PID with unique prime ideal (x), so is a DVR by the first definition, Conversely, if R is a DVR with nonzero prime ideal $P = (x)$, then I have already observed that every $y \in R$ is $x^n u$ for a unique unit u. Setting $v(y) = n$ and extending v to K^* by decreeing that $v(y^{-1}) = -v(y)$, it is easy to check that v is a valuation on K^* with valuation ring R. Thus DVRs are exactly the valuation rings of fields. The generator x of the prime ideal P is called a uniformizing or local parameter for R (p. 756).

Next I show how to construct valuation rings without using valuations. Given any PID R and a prime $p \in R$, let R_p be the subring of the quotient field K of R consisting of all fractions a/b with $p \not| b$. It is easy to check that R_p is indeed a subring containing R. Since R is a unique factorization domain, any $x \in K^*$ can be written as $\frac{p^n a}{b}$ for some $n \in \mathbb{Z}$ and relatively prime $a, b \in R$ with neither a nor b a multiple of p. Then the map v sending $\frac{p^n a}{b}$ to n is easily seen to be a valuation on K with valuation ring R_p.

In particular, R_p is Noetherian and integrally closed in its quotient field since any unique factorization domain has this property. Thus R_p is also a Dedekind domain. In fact the same properties follow from a weaker hypothesis.

Proposition; see Theorem 7, p. 757

Let R be a local Noetherian integral domain whose maximal ideal $M = (t)$ is principal. Then R is a PID (and thus also a DVR).

Proof.

Let $M_0 = \cap_{i=1}^{\infty} M^i$. Then $M_0 = MM_0$ and M_0 is finitely generated; since R is local with maximal ideal M, it follows from Nakayama's Lemma (proved last time) that $M_0 = 0$. If I is a nonzero proper ideal of R than there is $n \in \mathbb{N}$ with $I \subseteq M^n, I \not\subseteq M^{n+1}$. Letting $a \in I, a \notin M^{n+1}$ we have $a = t^n u$ for some $u \in R$, which must be a unit since $u \notin M$. But then $(a) = (t^n) = M^n$ and $I = (t^n)$ is principal, as desired. $\square$

With a little more work it can be shown that any Noetherian integrally closed integral domain with exactly two prime ideals is such that its maximal ideal is principal (Theorem 7, p. 757, part 5), so that any such ring is a DVR.

The two simplest examples of DVRs arise from the two simplest PIDs, namely $R = \mathbb{Z}$ and $R = K[x]$ with K a field. Letting p be a prime number in the first case and the polynomial x in the second, we get these examples. The construction of R_p from R is called localization (since it produces a local ring). It goes far beyond the setting of PIDs and prime elements, playing a crucial role in commutative algebra (as you will see next term).

I close by mentioning that there are many valuations that are not discrete and consequently many valuation rings that are not DVRs. Let T be a totally ordered additive abelian group, so that T is both an abelian group and a totally ordered set such that if $t_1 \leq u_1$ and $t_2 \leq u_2$ with $t_i, u_i \in T$ then $t_1 + u_1 \leq t_2 + u_2$. Define a (T-valued) valuation on a field K to be a map $v : K^* \to T$ satisfying the above properties of a discrete valuation. The subring R of K consisting of all $k \in K^*$ with $v(k) \geq 0$ together with 0 is such that for every $k \in K^*$ either $k \in R$ or $k^{-1} \in R$ (or both). It is called the valuation ring of K and v. The group T is called the value group of v.

Conversely, given a valuation ring R of a field K (containing either k or k^{-1} for any $k \in K^*$), we have the quotient multiplicative group $G = K^*/U$, where U is the group of units in R. This group is totally ordered via the rule $g \geq h$ if $gh^{-1} \in R$. The map sending any $k \in K^*$ to its image in G is then a G-valued valuation on K.with valuation ring R.

Moreover, given any totally ordered abelian group T (written multiplicatively), let L be any field. Form the group algebra LT, consisting of all finite formal linear combinations $\sum \ell_i t_i$ with $\ell_i \in L, t_i \in T$ and the t_i distinct. This ring is an integral domain; to see this, let $x = \sum \ell_i t_i, y = \sum m_j s_j$ be two elements of it with all the ℓ_i and m_j nonzero. Order the terms so that t_1 is the $\leq$-smallest of the t_i and s_1 the $\leq$-smallest of the s_j. Then $t_1 s_1$ the unique $\leq$-smallest element of T arising in the product xy, with nonzero coefficient $\ell_1 m_1$, so that $xy \neq 0$. Thus LT has a quotient field K.

One can now define a T-valued valuation v on the nonzero elements of LT via $v(\sum \ell_i t_i) = t_1$, where the terms of the sum are arranged so that t_1 is the $\leq$-smallest term appearing with $\ell_1 \neq 0$. Extend v to K^* via $v(x^{-1}) = v(x)^{-1}$. Then one checks that v is indeed a valuation with value group T (but we do *not* recover LT from K as the valuation ring). Thus any totally ordered group is the value group of some valuation.

Next time I will show how to enlarge a valuation ring to something called its completion; this is an algebraic analogue of the completion of a metric space with very nice properties.