Lecture 3-3: Commutative Artinian rings

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I begin with a few additional remarks about semisimple Artinian rings and then turn to commutative Artinian rings, following section 16.1 of Dummit and Foote.

I observed last time that the group algebra KG of a finite group over a field K is Artinian. Given Maschke's Theorem, one would expect that KG is semisimple Artinian if the characteristic of K does not divide the order n of G. This is indeed the case. Any nilpotent ideal I of KG would have to act on KG via left multiplication by nilpotent matrices, which have trace 0. But now the trace of left multiplication by any $g \neq 1$ in G is 0 (since it sends $h \in G$ to $gh \neq h$), while the trace of left multiplication by 1 is n, which is nonzero in K under the hypothesis. Given a nonzero combination $\sum k_g g$ lying in I with $k_g \neq 0$, multiplication by g^{-1} yields another such combination in which the coefficient k_1 of 1 is nonzero, a contradiction.

Thus one gets an alternative proof of Maschke's Theorem (given the general results about semisimple Artinian rings proved last time). In fact, this was the first proof that I saw of this result as a graduate student; I learned the one involving averaging over the group only years later. Given the results of last time and earlier ones about division rings over the reals, one also sees that the real group algebra $\mathbb{R}G$ of a finite group G is isomorphic to a sum of matrix rings over $\mathbb{R}, \mathbb{C}$, and the quaternion ring $\mathbb{H}$. The simplest example of a group G where $\mathbb{H}$ arises in this algebra is naturally enough the group Q of quaternion units.

Now let R be a commutative Artinian ring. The theory of such rings is very similar to that of proper quotients of Dedekind domains, which arose earlier in conjunction with rings of integers over number fields. The first result is

Proposition

A commutative Artininan ring R has only finitely many maximal ideals.

Among all finite intersections of maximal ideals in R there is a minimal element $I = \cap_{i=1}^{n} M_i$. Then any maximal ideal M contains I by minimality, when M contains the product $M_1 \ldots M_n$, and then by primeness M_i for some i. This forces $M = M_i$.

Recall from last time that any irreducible (left) R-module takes the form R/L for a maximal left ideal L, which for commutative rings is the same thing as a maximal ideal. Thus the Jacobson radical J of R, being the intersection of the kernels of these modules, is just the intersection of the maximal ideals of R. If these distinct maximal ideals are $M_1, \ldots, M_n$, then $M_i + M_j = R$ for $i \neq j$, whence by the Chinese Remainder Theorem we get $J = \cap M_i = \prod M_i$ and $R/Jcong \oplus_{i=1}^n R/M_i$.

In particular, if R is semisimple in the sense that $J = 0$ (this is the general definition of semisimplicity for any ring, even a non-Artinian one) one sees that $J = 0$ and R is a direct sum of fields R/M_i; this is Theorem 3 on p. 752. For a general Artinian R, we saw last time that $J^m = 0$ for some m, whence $\cap M_i^m = 0$ and the Chinese Remainder Theorem implies that $R \cong \oplus_{i=1}^m R/M_i^m$. Since the only maximal ideal of R containing M_i^m for any m is M_i itself, it also follows that each quotient R/M_i^m is local; that is, it has exactly one maximal ideal. You will be seeing a lot of local rings next quarter.

Thus a commutative Artinian ring is a finite direct product of Artinian local rings. Given a commutative local Artinian ring R there is n with $x^n = 0$ for all x in the unique maximal ideal M, while if x does not lie in M then it is a unit in R (the unique maximal ideal M contains all the nonunits in R). But now even more can be said: any quotient M^i/M^{i+1} is naturally an R/M-module, that is a vector space over $K = R/M$ and its submodules are the same as its subspaces over K. The Artinian condition then guarantees that M^i/M^{i+1} has finite dimension over K. Since this is true for all i and $M^n = 0$, it follows that every quotient M^i/M^{i+1} and R itself satisfy the *ascending* chain condition on ideals.

Thus a commutative Artinian local ring is necessarily Noetherian (in the terminology introduced last time). Since a general commutative Artinian ring is a finite direct product of Artinian local rings, any commutative Artinian ring is Noetherian. It is also easy to show that any finitely generated module M over a commutative Artinian ring has finite length; that is, there is an integer n such that any strictly increasing chain $M_0 \subset M_1 \cdots \subset M_m$ of submodules of M has $m \leq n$. I will use this fact next quarter.

A further property is that in any commutative Artinian ring R every prime ideal is maximal. To see this assume first that R is local with maximal ideal M. Then $M^n = 0$ for some n, whence any prime ideal P contains M^n and so must equal M. In general, R is a finite direct product of Artinian local rings R_i, each with maximal ideal M_i. A prime ideal P of R must intersect each R_i in a prime ideal, and hence its unique maximal ideal, unless it contains R_i. Since R/P is an integral domain, we must in fact $P \supset R_i$ for all but one i and $P \cap R_i = M_i$ for the exceptional i. Thus P is maximal, as claimed.

We have seen that a commutative Artinian ring R is necessarily Noetherian and every prime ideal of R is maximal. Remarkably enough, the converse is true: a commutative ring R is Artinian if and only if it is Noetherian and every prime ideal is maximal. To prove this, let R be a commutative Noetherian ring and suppose that R is not Artinian. I must show that there is a prime ideal P of R that is not maximal. Let $\mathcal{S}$ be the set of ideals I of R such that R/I is not Artinian. Then $\mathcal{S}$ is not empty since it contains the 0 ideal. Let Q be a maximal element of $\mathcal{S}$, which exists since R is Noetherian.

Now I claim that $S = R/Q$ is an integral domain, so that Q is prime; since fields are trivially Artinian, $P = Q$ furnishes the desired example of a prime ideal that is not maximal. Choose a nonzero $a \in S$, let I be the annihilator of a (consisting of all $b \in S$ with $ba = 0$) and consider the short exact sequence $0 \to S/I \to S \to S/(a) \to 0$ of S-modules, where the map from S/I to S is multiplication by a (which is well defined). If $I \neq 0$ then S/I and $S/(a)$ are proper quotients of S, forcing them to be Artinian by the maximality of Q; but then an easy formal argument shows that S is Artinian as well, a contradiction. Hence $I = 0$, so that S is an integral domain and Q is prime, as desired. It is not maximal since S/Q is not Artinian.

A further fact which we will have many occasions to use next quarter is the following one.

Nakayama's Lemma: Proposition 1, p. 751

Let M be a finitely generated module over the commutative ring R with Jacobson radical J s such that $JM = M$. Then $M = 0$.

If $M \neq 0$ then let $m_1, \ldots, m_n$ be a minimal set of generators of M. Then we can write $m_n = \sum_{i=1}^{n} j_i m_i$ for some $j_i \in J$ since $JM = M$. We saw last time that $1 - j_n$ is necessarily a unit in R, so we can write both $(1 - j_n)m_n$ and m_n as a combination of $m_1, \ldots, m_{n-1}$. This contradicts the minimality of the m_i.

This proof works over a noncommutative ring as well, but in practice the result is seldom applied except in the commutative setting.