Lecture 2-5: Cohomology of finite groups

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I begin a new unit on group cohomology, following Chapter 17 of the text. In effect I will redo representation theory in a different context, namely that of $\mathbb{Z}$-modules rather than complex vector spaces.

Given a finite group G, a G-module will now be an abelian group A on which G acts linearly (see p. 798). The analogue of the complex group algebra $\mathbb{C}G$ is then the integral group ring $\mathbb{Z}G$, consisting of all integral combinations $\sum_{g \in G} z_g g$ with $z_g \in \mathbb{Z}$ (see p. 237). Clearly a G-module by the above definition is the same thing as a $\mathbb{Z}G$-module. This time, however, nothing like Maschke's Theorem holds, for over $\mathbb{Z}$ the only irreducible modules are cyclic groups of prime order; finite direct sums of these account for only a small fraction even of finite $\mathbb{Z}$-modules. Accordingly, I will not attempt to study the structure of $\mathbb{Z}G$ itself.

Instead I will focus on how a certain very particular $\mathbb{Z}G$-module, namely the trivial module $\mathbb{Z}$ (on which G acts trivially) and how it fits inside a larger $\mathbb{Z}G$-module. More precisely, let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathbb{Z}G$-modules. This induces an exact sequence $0 \to^G \to B^G \to C^G$ of their subgroups of elements fixed by G, corresponding to the exact sequence $0 \to \hom_G(\mathbb{Z}, A) \to \hom_G(\mathbb{Z}, B) \to \hom_G(\mathbb{Z}, C)$, but in general the map from B^G to C^G is not surjective, since $\mathbb{Z}$ is not projective as a $\mathbb{Z}G$-module.

We therefore define $H^n(G, A$, the nth cohomology group of G with coefficients in A, to be the Ext group $\mathrm{Ext}_{\mathbb{Z}G}(\mathbb{Z}, A)$ defined last quarter. See the last sentence in the paragraph following equation 17.17 in the text on p. 799; the text gives a different definition of $H^n(G, A)$, which I will get to shortly. Thus $H^0(G, A) = A^G$, the subgroup of G-fixed elements, while the groups $H_n(G, A)$ for $n > 0$ may be viewed as the images of higher derived functors of the functor taking A to A^G. We can compute $H^n(G, A)$ by projectively resolving the trivial module $\mathbb{Z}$ over $\mathbb{Z}G$, taking homomorphisms of each term into A, and then computing the cohomology of the resulting cochain complex.

If for example $G = \mathbb{Z}_n$ is cyclic of order n, then $\mathbb{Z}G$ identifies with the quotient $\mathbb{Z}[x]/(x^n - 1)$. Last quarter I computed the groups $\mathrm{Ext}^i_{\mathbb{Z}_n}(\mathbb{Z}_m, D)$, where m, n are positive integers with $m|n$ and D is a $\mathbb{Z}_n$-module; here $\mathbb{Z}_m$ is regarded as a $\mathbb{Z}_n$-module via the natural surjection $\mathbb{Z}_n \to \mathbb{Z}_m$.

Adapting the projective resolution of $\mathbb{Z}_m$ used to make that computation, one finds that there is a periodic projective resolution $\{P_i\}$ of $\mathbb{Z}$ in which $P_i = \mathbb{Z}[x]/(x^n - 1)$ for all i. The map from P_0 to $\mathbb{Z}$ is the quotient map by the ideal $(x - 1)$; the maps $P_{2m+1} \to P_{2m}$ are all multiplication by $x - 1$, while the maps $P_{2m} \to P_{2m-1}$ are all multiplication by $N = 1 + x + \ldots + x^{n-1}$ for $m \geq 1$.

Accordingly, we have

Example, p. 801

With notation as above, we have

$$H^n(G,A) = \begin{cases} A^G \text{ if } n = 0 \\ {}_N A/(x-1)A \text{ if n is odd} \\ A^G/NA \text{ if n is even }, n \geq 2 \end{cases}$$

where ${}_N A$ denotes $\{a \in A : NA = 0\}$. For general groups G, it turns out that there is a uniform way to resolve $\mathbb{Z}$ by free modules. Write $R = \mathbb{Z}G$ and for $n > 0$ let F_n be the $(n+1)$-fold tensor power $\otimes^{n+1} R$, the tensor products taking place over $\mathbb{Z}$. Here G acts on tensors by left multiplication on the first factor. Put $F_{-1} = \mathbb{Z}$. It is easy to check that F_n is free of rank $|G|^n$ over R for $n > 0$, a basis being given by the products $1 \otimes g_1 \otimes \cdots \otimes g_n$ as the g_i range over G.

Define the boundary operator $d_n : F_n \to F_{n-1}$ via

$$d_n(g_0 \otimes \cdots \otimes g_n) = \sum_{i=0}^{n-1} (-1)^i g_0 \otimes \cdots g_i g_{i+1} \otimes \cdots \otimes g_n + (-1)^n g_0 \otimes \cdots \otimes g_{n-1}$$

for $n > 0$, while $d_0(g) = 1$. An easy calculation shows that $d_{n-1} d_n = 0$. In more detail, there are four kinds of terms arising in $d_{n-1} d_n$: those involving a product $g_i g_{i+1} g_{i+2}$ of three successive group elements; those involving two products $g_i g_{i+1}, g_j g_{j+1}$ of two successive elements; those involving one product of two successive g_i and omitting g_n; and one term omitting both g_{n-1} and g_n. In all four cases, there are two cancelling contributions to each term; so $d_{n-1} d_n = 0$.

We also have a map $s_n : F_n \to F_{n+1}$ sending $g_0 \otimes \cdots \otimes g_n$ to $1 \otimes g_0 \otimes \cdots \otimes g_n$. An even easier calculation than the one on the last slide shows that $d_{n+1}s_n + s_{n-1}d_n = 1$, the identity map, whence if $x \in F_n$ lies in the kernel of d_n, then $x = d_{n+1}s_n x$ also lies in the image of d_{n+1}. Hence the F_n and d_n above define a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module (called the bar resolution (p. 799)).

Taking homomorphisms into A, we see that if we denote by $C^n(G, A)$ the set of maps from the Cartesian product G^n of n copies of G to A and if we define a new map $d_n : C^n(G, A) \to C^{n+1}(G, A)$ via

$$d_n f(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1})$$

$$+\sum_{i=1}^{n}(-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n)$$

$$+(-1)^{n+1} f(g_1, \ldots, g_n)$$

then $d_n d_{n-1} = 0$, so we can form the quotient $\ker d_{n-1} / \operatorname{im}(d_n)$. The text defines $H^n(G, A)$ as this quotient (p. 800).

In practice, the formula on the previous slide is more useful for interpreting $H^n(G, A)$ than for computing it, though one immediate consequence is that if $mA = 0$ for some $m \in Z$ then $mH^n(G, A) = 0$ for all n (Proposition 20, p. 801).

From the long exact sequence oof Ext groups arising from any short exact sequence of modules we get

Theorem 21, p. 802

Given a short exact sequence $0 \to A \to B \to C \to 0$ of G-modules, there is a long exact sequence

$$0 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to \cdots$$

A G-module M is called cohomologically trivial if $H^n(G, M) = 0$ for $n \geq 1$. It follows at once from the long exact sequence above that if $0 \to A \to M \to C \to 0$ is a short exact sequence of G-modules and M is cohomologically trivial then $H^{n+1}(G, A) \cong H^n(G, C)$ for all $n \geq 1$ (Corollary 22, p, 802). This last fact is often referred to as dimension shifting.

To produce examples of cohomologically trivial modules one needs a more general construction similar to the induced module construction of last quarter.

Definition, p. 803

If H is a subgroup of G and A is an H-module then the *coinduced module* $M = M_H^G(A)$ is defined to be $\hom_{\mathbb{Z}H}(\mathbb{Z}G, A)$; it consists of all maps $f : G \to A$ with $f(hx) = hf(x)$ for $x \in G, h \in H$. This becomes a G-module via the recipe $(g \cdot f)(x) = f(xg)$ for $f \in M, x, g, \in G$.

This definition makes sense for any group G and subgroup H. If H has finite index in G then it is not difficult to check that $M_H^G(A)$ coincides with the induced module $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$.

It turns out that the G-cohomology of the coinduced module $M_H^G(A)$ coincides with the H-cohomology of A. This is

Shapiro's Lemma, p. 804

For any subgroup H of G and any H-module A we have $H^n(G, M_h^G(A)) = H^n(H, A)$ for all $n \geq 0$.

Proof.

Since $\mathbb{Z}G$ is free over $\mathbb{Z}H$, any projective resolution $\{P_n\}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ is also a projective resolution over $\mathbb{Z}H$. Taking homomorphisms into A, we compute the cohomology group $H^n(H, A)$. Now we have a general isomorphism

$$\Phi : \hom_{\mathbb{Z}G}(P_n, \hom_{\mathbb{Z}H}(\mathbb{Z}G, A)) \cong \hom_{\mathbb{Z}H}(P_n, A)$$

given by $\Phi(f)(p) = f(p)(1)$ for $f \in \hom_{\mathbb{Z}G}(P_n, \hom_{\mathbb{Z}H}(\mathbb{Z}G, A)), p \in P_n$. The inverse map Ψ has $\Psi(f)(p)(g) = f(gp)$, which makes sense since P_n is a G-module. The isomorphism commutes with the cochain maps, so defines an isomorphism of the cohomology groups, as desired. □

Coinducing from the trivial subgroup, we deduce that $M_1^G(A)$ is cohomologically trivial for any abelian group A.