Lecture 2-28: Artin-Wedderburn theory

February 28, 2025

Returning to noncommutative rings, I will treat a topic omitted from my coverage of representation theory last term, namely the theory of semisimple Artinian rings (covered in section 18.2 of Dummit and Foote, though my treatment will be somewhat different). Semisimple Artinian rings include group algebras over any field of characteristic 0, not just $\mathbb{C}$, and many other examples.

The basic definition is

Definition

A (noncommutative) ring R is called *(left) Artinian* if every descending chain $L_1 \supset L_2 \supset \ldots$ of left ideals of R stabilizes, so that $L_n = L_{n+1} = \ldots$ for some n; equivalently, if every nonempty collection of left ideals has a minimal element. We say that R is *semisimple Artinian* if in addition there are no nonzero nilpotent two-sided ideals I of R, that is, ideals I such that $I^k = 0$ for some k.

Note that Dummit and Foote use the phrase "with minimum condition" instead of Artinian, reserving the Artinian terminology for commutative rings (see sections 16.1 and 16.2). I will show shortly that semisimple left Artinian rings are the same as semisimple right ones. For now observe that every finite-dimensional algebra A over a field F and in particular any (finite) group algebra over a field) is automatically left and right Artinian.

For future reference I note that the parallel condition (that every *ascending* chain of left ideals stabilizes, or that every nonempty collection of left ideals has a maximal element) also has a name; rings satisfying it are called left Noetherian, or just Noetherian if they are commutative. Next term you will see that for commutative rings the Noetherian condition is exactly what is needed to make a lot of theorems work; though it is a strong one, there is a huge variety of important and interesting rings satisfying it. By contrast, Artinian rings (both commutative and noncommutative) are much more specialized; in fact every left Artinian ring turns out to be left Noetherian as well.

The notion of irreducibility of a module, which is not very useful for general rings, turns out to play a key role for Artinian ones. Recall some terminology from last quarter: given a left R-module M, its endomorphism ring, denoted $\mathrm{End}_R M$, consists of the R-module maps from M to itself. In homework last quarter, you proved Schur's Lemma in its general context: it asserts that $\mathrm{End}_R M$ is a division ring if M is an irreducible R-module.

The first main result is the following. Recall that a ring R is called simple if it has no nonzero proper two-sided ideals. A simple commutative ring is a field, but a simple noncommutative ring need not be a division ring.

Theorem

Any simple left or right Artinian ring R is isomorphic to $M_n(D)$, the ring of $n \times n$ matrices over a division ring D for a unique n and D, up to isomorphism. The only irreducible left (or right) R-module M is the space D^n of column (or row) vectors over D of length n. Every R-module is a direct sum of copies of D^n.

Proof.

This will take several steps. First let L be a minimal nonzero left (or right) deal of R; such exists because R is Artinan. Then L is an irreducible left R-module, so certainly R has such a module. Given any irreducible left module M, its annihilator $A(M)$, consisting of all $r \in R$ with $rM = 0$, is a two-sided ideal of R and so must be 0. In particular there is $m \in M$ with $Lm \neq 0$; but then Lm is a submodule of M, which must be all of M by irreducibility. Moreover, the map $L \to M$ sending x to xm has kernel an ideal strictly contained in L, whence this kernel must be 0 and $M \cong L$ as a left R-module. In particular, any two irreducible left R-modules are isomorphic. $\square$

Proof.

Letting $D = \text{End}\, M$, we have seen that D is a division ring. I now claim that given any $m_1, \ldots, m_n \in M$ linearly independent over D and any $x_1, \ldots, x_n \in M$ there is $r \in R$ with $rm_i = x_i$ for all i. This is called the density condition. It is proved by induction on n; it is clear for $n = 1$ (by irreducibility). If it holds for n and $m_1, \ldots, m_{n+1}$ are independent over D, then the range $N_{n+1} = R(m_1, \ldots, m_{n+1})$ of the action of R on the m_i has tuples with any set of first n coordinates. If the intersection of N_{n+1} with $\{(0, \ldots, 0, m) : m \in M\}$ is 0, then for every $x_1, \ldots, x_n \in M$ there is a unique $x_{n+1} \in M$ with $(x_1, \ldots, x_n, x_{n+1}) \in N_{n+1}$ and the map sending $(x_1, \ldots, x_n)$ to x_{n+1} is an R-module map from M^n to M. $\square$

Proof.

Applying Schur's Lemma to each coordinate, one sees that there are $d_1, \ldots, d_n \in D$ with $x_{n+1} = \sum_{i=1}^{n} x_i d_i$; here I am regarding M as a *right* vector space over D. But this is not possible, since $(m_1, \ldots, m_{n+1}) \in N_{n+1}$ and the m_i are independent over D. Hence N_{n+1} contains $0 \times \ldots \times 0 \times M$, by irreduciblity and is all of M^{n+1}, as claimed. Now if M had infinitely many elements $m_1, m_2, \ldots$ independent over D, then the left ideals L_i of R consisting of all x with $xm_j = 0$ for all $j \leq i$ would form a strictly decreasing chain, which is impossible. Hence $M \cong D^n$ must be finite-dimensional over D and R must act on M by the full set $M_n(D)$ of matrices on column vectors, this action commuting with the *right* D-action on the vectors Moreover R itself is a direct sum of n copies of D^n (one for each of the n columns), so D^n is projective over R and every R-module is a direct sum of copies of D^n. $\square$

Proof.

Finally, the kernel of the R-action on M is trivial, so in fact $R \cong M_n(D)$. The ring D is the endomorphism ring of the unique irreducible R-module up to isomorphism, while n is uniquely determined as the dimension of this module over D. $\square$

Conversely, any ring $M_n(D)$ with D a division ring is simple (left and right) Artinian: it is finite-dimensional over D, so satisfies the descending chain condition on left or right ideals and is simple because any proper two-sided ideal I would have to act irreducibly on D^n and satisfy the density condition, whence $I = M_n(D)$. I mention that it is possible for two nonisomorphic nondivision rings R, S to satisfy $M_n(R) \cong M_n(S)$ for some n.

What about nonsimple Artinian rings? To discuss these I need a general definition.

Definition

The *Jacobson radical* J of any ring R consists of all $x \in R$ with $xM = 0$ for all irreducible left R-modules M.

Clearly J is a two-sided ideal of R. We have the following handy criterion for an element to lie in J.

Lemma

For $x \in R$ we have $x \in J$ if and only if $1 - yx$ has a left multiplicative inverse in R for all $y \in R$.

Proof.

An irreducible left R-module M is generated by any nonzero element m; so any such module takes the from R/L for L a maximal left ideal of R. If $x \in R$ fails to lie in L, so that it does not lie In the annihilator of R/L, then $L + Rx$ is a left ideal properly containing L and so must be all of R, whence $m + yx = 1$ for some $y \in Rm \in L$ and $m = 1 - yx \in L$ is not left invertible. Conversely, if $1 - yx$ is not left invertible for some $y \in R$, then it lies in a proper left ideal, which can be enlarged by Zorn's Lemma to a maximal left ideal L, which cannot contain x. Then x does not lie in the annihilator of R/L. $\square$

The reason for the above definition of semisimplicity emerges from the following result.

Theorem

If R is left or right Artinian, then J is nilpotent. R is semisimple if and only if $J = 0$.

The descending chain $J \supset J^2 \supset \ldots$ must stabilize, say at $P = J^n$. If $P = P^2 \neq 0$ then choose a minimal left ideal I with $PI = P^2 I \neq 0$ and choose a nonzero $x \in I$ with $Px \neq 0$. Then Px is another left ideal with $P(Px) = Px \neq 0$, so by minimality we must have $Px = Rx$, so that $x = zx, (1 - z)x = 0$ for some $z \in P$. But then $1 - z$ is left invertible by the previous result, whence $x = 0$, a contradiction. Finally, if $J = 0$ and N is a nilpotent ideal of R, then for any $x \in N, y \in R$ we have $(yx)^k = 0$ for some k, whence $1 - yx$ has the left inverse $1 - yx + (yx)^2 - \ldots \pm (yx)^{k-1}$, forcing $x \in J$, so 0 is the only nilpotent ideal of R.

The classification theorem for semisimple Artinian rings then reads

Theorem 4, p. 854

The ring R is semisimple left or right Artinian if and only if it is a finite direct sum $\oplus M_{n_i}(D_i)$ of matrix rings over division rings D_i. In this case every irreducible left or right R-module is isomorphic to $D_i^{n_i}$ for some i and every left or right R-module is a direct sum of copies of such modules.

Proof.

Any such R has a minimal left ideal and so an irreducible left module M_1. The quotient R/K_1 of R by the kernel of the action on M is then a matrix ring $M_{n_1}(D_1)$ for some division ring D_1 by the first main result. If $K_1 \neq 0$ then R/K_1 has an irreducible left module M_2 and $R/(K_1 + K_2)$ is a matrix ring $M_{n_2}(D_2)$ for some D_2. Continuing in this way, we get irreducible left modules $M_1, M_2, \ldots$ for R; letting I_i be the kernel of its action on $\oplus_{j=1}^{i} M_j$ the chain $I_1 \supset I_2 \supset \ldots$ stabilizes, necessarily at 0, since $J = 0$. The density condition implies the desired decomposition of R; since R is a direct sum of simple left or right ideals, any irreducible R-module is isomorphic to one of them and is projective, so that every module is a direct sum of simple one-sided ideals. □