Lecture 2-26: Finitely generated modules over Dedekind domains

February 26, 2025

The goal for this lecture is to classify finitely generated modules over a Dedekind domain; this classification is analogous to but somewhat more complicated than the corresponding one for PIDs. Let R be a Dedekind domain with quotient field K.

I begin by looking at ideals of R, regarded as modules.

Proposition

Two ideals I, J of R are isomorphic as R-modules if and only if the classes $[I], [J]$ are equal, so that $J = kI$ for some $k \in K^*$.

If $J = kI$, then multiplication by k is an isomorphism from I to J; conversely, given isomorphism $\pi : I \to J$ and a nonzero $i \in I$, the image $\pi(i)$ must be ki for a unique $k \in K^*$, whence $\pi(ri) = kri$ for all $r \in R$, whence $\pi(xi) = kxi$ for all $x \in K$ such that $kx \in I$, since for all $y \in I$ and $s \in R^*$, if $y/s \in I$, then it is the only element z of I with $sz = y$. Hence $J = kI$ in this case.

Next I investigate when two direct sums of ideals are isomorphic. To do this I need the following lemma, given as an exercise in Dummit and Foote.

Lemma; see Exercise 12, p. 774

Given nonzero ideals I, J of R there is $c \in K^*$ with cI, J coprime ideals (so that $cI + J = R$).

Choose $a \neq 0$ in I. Then $I' = I^{-1}(a)$ is an ideal of R with $II' = (a)$. The quotient $R' = R/I'J$ is principal ideal ring, as shown last time, so there is $\bar{b} \in R'$ with $I'/I'J = (\bar{b})$; letting $b \in R$ be a preimage of $\bar{b}$, we then have $I' = I'J + bR$. Multiplying this last equation by I and dividing it by a, we get $cI + J = R$ with $c = b/a$ and cI an ideal of R, as desired.

The basic computation is then

Corollary; see p. 770

For any nonzero fractional ideals I, J of R we have $I \oplus J \cong R \oplus IJ$ as R-modules.

Replacing I, J by an isomorphic ideals, we may assume that $I + J = R$, whence $I \cap J = IJ$. The map $\pi : I \oplus J \to R$ seding (i, j) to $i + j$ is surjective with kernel $I \cap J = IJ$. Since R is projective as an R-module, the exact sequence $0 \to IJ \to I \oplus J \to R \to 0$ spilts, so that $I \oplus J \cong R \oplus IJ$, as claimed. Iterating this last result, we get $I_1 \oplus \cdots \oplus I_n \cong R^{n-1} \oplus I_1 \cdots I_n$ for any nonzero ideals I_i. We also see that any nonzero ideal I of R is projective, being a direct summand of $I \oplus I^{-1} \cong R \oplus R$.

The classification result for direct sums of ideals then reads

Proposition 21, p. 769

Given nonzero ideals $I_1, \ldots, I_n, J_1, \ldots, J_m$ of R we have $\oplus I_i \cong \oplus J_j$ if and only if $n = m$ and $[I_1 \ldots, I_n] = [J_1 \ldots J_m]$.

Proof.

We know that the left side is isomorphic to $S = R^{n-1} \oplus I = R^{n-1} \oplus I_1 \dots I_n$ and the right side to $T = R^{m-1} \oplus J = R^{m-1} oplus J_1 \dots J_m$, so the sufficiency of the condition is clear. Defining the rank of a finitely generated R-module to be the maximum number of elements of it linearly independent over R, we see that the rank of S is n and the rank of T is m, so $S \cong T$ forces $n = m$. Finally, an isomorphism from S to T would have to be given by multiplication by an $n \times n$ matrix A over K with determinant $d \in K^*$; similarly the inverse isomorphism from T to S would be given by multiplication by A^{-1}, regarding the elements of S and T as column vectors in R^n. □

Proof.

The proof of the Cayley-Hamilton Theorem for matrices over commutative rings shows that AS contains dS and likewise $A^{-1}T$ contains $d^{-1}T$, forcing $dI \subset J$, $d^{-1}J \subset I$. But then $dI = J$, $d^{-1}J = I$, forcing $[I] = [J]$, as claimed. □

The classification of finitely generated torsion-free modules over R is then completed by the following result. Recall that the torsion submodule T of a module M over an integral domain A consists of the $m \in M$ for which there is nonzero $a \in A$ such that $am = 0$. The module M is called torsion-free if its torsion submodule T is 0.

Proposition 21 again

Any finitely generated torsion-free R-module is isomorphic to a finite direct sum of ideals of R.

Proof.

I showed last quarter that given any $\mathbb{Z}$-module M, the kernel of the natural map from M to $M' = M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the torsion submodule of M; this is because M' can be modelled as the set of ordered pairs $(s, m) \in \mathbb{Z}^* \times M$ modulo the equivalence relation $(s, m) \sim (t, n)$ if there is $u \in \mathbb{Z}^*$ with $u(sn - tm) = 0$. This calculation carries over to R-modules, replacing $\mathbb{Z}$ by R and $\mathbb{Q}$ by K. Given a finitely generated torsion-free R-module M, then, I can assume that it is a finitely generated submodule of K^n for some n. I show that any such module is isomorphic to a direct sum of ideals by induction on n, the case $n = 1$ being clear by the definition of fractional ideal. If the result holds for n and M is a submodule of K^{n+1}, then projection to the last coordinate maps M onto a submodule of K, which is a fractional ideal I. □

Proof.

Since I is projective (or 0), M is the direct sum of I and the intersection $M \cap K^n$, which is isomorphic to a direct sum of finitely many ideals; so M is also. □

Thus a finitely generated module is torsion-free if and only if it is projective. The distinction between freeness and projectivity for R-modules is measured exactly by the class group of R.

At the other extreme, we have a classification of torsion modules which is essentially the same as that of torsion modules over a PID.

Theorem 22, p. 771

A finitely generated R-module T is torsion if and only if it is a finite direct sum of proper quotients R/I_i of R, or if and only if it is the finite direct sum of quotients of R by powers of prime ideals.

Any finitely generated module is a quotient of R^n for some n, so it suffices to classify the submodules of R^n. The proof of the previous theorem shows that these are exactly the direct sums of at most n ideals of R. We have already seen (by the Chinese Remainder Theorem) that any quotient R/I is isomorphic to the direct sum $\oplus_{i=1}^m R/P_i^{n_i}$, where $P_1^{n_1} \dots P_m^{n_m}$ is the prime factorization of I.

The analogue of the elementary divisor version of the classification of finitely generated modules over a PID asserts that every finitely generated torsion module over R is isomorphic to a direct sum $\oplus_{i=1}^{m} R/I_i$, where $I_1 \subset \cdots \subset I_m$.

Finally, given an arbitrary finitely generated R-module M, let T be its torsion submodule, so that $M/T = N$ is torsion-free and finitely generated (since M is). The short exact sequence $0 \to T \to M \to N \to 0$ splits, since N is projective, so $M \cong T \oplus N$. Finally, $T \cong M/N$ is finitely generated since M is. Now we finally get the full classification theorem.

Theorem 22 again

Any finitely generated R-module M is isomorphic to $R^n \oplus I \oplus \oplus_{i=1}^r R/P_i^{n_i}$ for some nonnegative integer n, ideal I, and powers $P_i^{n_i}$ of primes ideals P_i. Two such direct sums $R^n \oplus I \oplus \oplus_{i=1}^r R/P_i^{n_i}, R^m \oplus J \oplus \oplus_{i=1}^s R/Q_i^{m_i}$ are isomorphic if and only if $m = n, [I] = [J], r = s$, and the $P_i^{n_i}$ are a rearrangement of the $Q_j^{m_j}$.

Proof.

We have already seen that M must take the desired form. Given two such decompositions, the above uniqueness result for the quotient M/T of M by its torsion submodule T shows that $n = m$ and $[I] = [J]$. The proof of the uniqueness of the powers of the prime ideals proceeds as in the PID case, looking at the dimensions of the various vector spaces $P^n T/P^{n+1} T$ over R/P as P runs over the nonzero prime ideals of R and n runs over the nonnegative integers. □