# Lecture 2-14: Group homology

February 14, 2025

I wrap up this unit of the course with an account of the homology groups $H_n(G, A)$ of a finite group $G$ with coefficients in a $G$-module $A$, Although these are not generally as useful as the cohomology groups, they too can give important information about $G$. There are also many interesting parallels between homology and cohomology.

Let $G$ be a finite group and $A$ a $G$-module.

## Definition

The *homology group* $H_n(G, A)$ is defined to be $\mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A)$; that is the $n$th higher derived functor of the tensor product functor $- \otimes_{\mathbb{Z}G} \mathbb{Z}$; here $\mathbb{Z}$ is regarded as a trivial $\mathbb{Z}G$-module.

In particular $H^0(G, A) = A \otimes_{\mathbb{Z}G} \mathbb{Z} = A_G = A/S$, where $S$ is the subgroup of $A$ spanned by the elements $(g-1)a$ as $g$ runs over $G$ and $a$ over $A$. We can also write $S$ as $IA$, where $I$ is the augmentation ideal of $\mathbb{Z}G$ spanned by all differences $g - 1$.

A standard first example (as with cohomology) is the case of a cyclic group. As a change of pace take $G = \mathbb{Z}$ to be infinite cyclic. We can identify $\mathbb{Z}G$ with the ring $L$ of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ familiar from complex analysis. We have the augmentation map from $L$ to $\mathbb{Z}$ sending all powers $t^n$ to 1. Coupled with multiplication by $t - 1$, regarded as a map from $L$ to itself, we get a finite projective resolution of $L$. Tensoring with $A$, we get

$$H_n(\mathbb{Z}, A) = H^n(\mathbb{Z}, A) = 0, n \neq 0, 1$$

$$H_1(\mathbb{Z}, A) = H^0(\mathbb{Z}, A) = A^{\mathbb{Z}}; H^1(\mathbb{Z}, A) = H_0(\mathbb{Z}, A) = A_{\mathbb{Z}}$$

If instead $G = \mathbb{Z}_m$ is finite cyclic, say generated by $\sigma$, then

$$H_n(G,A) = \begin{cases} A_G & \text{if } n = 0; \\ A^G/NA & \text{if } n \text{ is odd}; \\ ({}_N A)_G & \text{if } n \text{ is even, } n \geq 2. \end{cases}$$

where $N = \sum_{g \in G} g$, the norm of $G$, and ${}_N A$ denotes the set of $a \in A$ with $Na = 0$.

$$H^n(G,A) = \begin{cases} A^G & \text{if } n = 0; \\ ({}_N A)_G & \text{if } n \text{ is odd}; \\ A^G/NA & \text{if } n \text{ is even, } n \geq 2. \end{cases}$$

In this case the resolution is periodic rather than finite.

Note the duality between homology and cohomology here. It turns out that a rather weak property of $A$ is enough to force all higher homology and cohomology groups to be 0, even for noncyclic groups. To state it, suppose that $G$ is finite with order $m$. The norm $N$ of $G$, defined in the last slide, is easily seen to satisfy $N^2 = mN$; also $N$ is central in $\mathbb{Z}G$. Letting $\mathbb{Z}'$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{m}$, we then have $e := \frac{1}{m}N \in \mathbb{Z}'G$ and $e$ is an idempotent in $\mathbb{Z}'G$, meaning by definition by $e^2 = e$. Given any ring $R$ with a central idempotent $e$, $R$ is the direct sum of the two-sided ideals $Re$ and $R(1-e)$ (direct because multiplication by $e$ is the identity on $Re$ but the zero map on $R(1-e)$).

Now we can state our theorem.

**Theorem**

With notation as above, if $A$ is a $G$-module such that multiplication by $m$ is an isomorphism from $A$ to itself, or equivalently if $A$ is a $\mathbb{Z}'$-module, then $H^n(G, A) = H_n(G, A) = 0$ if $n > 0$.

## Proof.

if $a \in A^G$, then $a = \frac{1}{m} Na = ea$, and conversely $eA \subset A^G$, so $A^G = eA$. Setting $R = \mathbb{Z}' G$, we find that the ideal $I = R(1-e)$ is the kernel of multiplication by $e$, whence $(1-e)A = (1-e)R \otimes_R A = IA$ and $A_G = A/IA = eA$. Since $eR$ is a direct summand of $R$, both $\mathrm{Tor}_n^R(eR, A)$ and $\mathrm{Ext}_R^n(eR, A)$ are 0 for $n > 0$. Applying something called flat base change for Tor (see §17.1 of Dummit and Foote), we get $\mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A) = \mathrm{Tor}_n^R(\mathbb{Z} \otimes R, A) = \mathrm{Tor}_n^R(eR, A) = 0$ for $n > 0$. The argument is similar for $H^n$, replacing Tor by Ext. $\square$

Thus all the homology or cohomology of a finite group arises from the failure of multiplication by $m$ to be an isomorphism. (More generally, if $G$ has order $m$, then we have $mH^n(G, A) = 0$ for all $n > 0$; we observed this earlier for cohomology.) I now turn attention to the first homology and cohomology groups, showing that these behave very differently for trivial and nontrivial actions. Consider first the case where $G$ acts trivially on $\mathbb{Z}$. There is a map $\theta$ from $G$ to $I/I^2$ ($I$ the augmentation ideal defined above) sending $g$ to $g - 1$; an easy calculation shows that $\theta$ is a homomorphism from $G$ under multiplication to $I/I^2$ under addition. Since $I/I^2$ is abelian, the commutator subgroup $[G, G]$ lies in the kernel of $\theta$. Similarly, we have a homomorphism $\sigma : I/I^2 \to G/[G, G]$ sending $g - 1$ to the coset of $g$ in $G/[G, G]$; this map sends $I^2$ to 1, so $\theta$ and $\sigma$ together given an isomorphism from $I/I^2$ to $G/[G, G]$.

**Theorem**

We have $H^1(G, \mathbb{Z} \cong G/[G, G]$.

## Proof.

From the short exact sequence $0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0$ and the associated long exact sequence in homology we get the exact sequence $\cdots \to H_1(G, \mathbb{Z}G) \to H_1(G, \mathbb{Z}) \to I_G \to (\mathbb{Z}G)_G \to \mathbb{Z} \to 0$. Since $\mathbb{Z}G$ is projective, $H^1(G, \mathbb{Z}G) = 0$ and the right-hand map is the isomorphism $(\mathbb{Z}G)_G \cong \mathbb{Z}G/I \cong \mathbb{Z}$ so $H_1(G, \mathbb{Z}) \cong I/I^2 \cong G/[G, G]$. $\square$

More generally, for any trivial $G$-module $A$, we have $H_0(G,A) = A, H_1(G,A) = G/[G,G] \otimes_{\mathbb{Z}} A$ (regarding the abelian group $G/[G,G]$ as a $\mathbb{Z}$-module); it turns out that $H_n(G,A) \cong H_n(G,\mathbb{Z}) \otimes A \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G,\mathbb{Z}), A)$. For cohomology, $H^1(G,A) \cong \hom_{\mathbb{Z}}(G/[G,G], A)$.

Hilbert's Theorem 90 that $H^1(G, L) = 0$ for a Galois group $G$ acting on the additive group of the Galois extension $L$ of a field $K$carries over to homology and in fact $H^n(G, L) = H_n(G, L) = 0$ for all $n \geq 1$. For the multiplicative group $L^*$, however, only the first cohomology group vanishes, not the higher ones or even the first homology group. (If $G$ is cyclic, then a calculation on a previous slide shows that $H_1(G, L^*) \cong K^*/N(L^*)$, where $N = N(L^*)$ denotes the group of norms of elements of $L^*$. We have seen already that we can have $K^* \neq N(L^*)$, e.g. if $K = \mathbb{R}, L = \mathbb{C}$.

Finally, I mention that while there is no homology analogue of the connection between extensions of groups and cohomology groups, the second homology group plays a crucial role in the theory of so-called universal central extensions of a group.

Next time I will shift gears, talking about the ring theory lying behind finite Galois extensions of $\mathbb{Q}$.