SOLUTIONS TO HOMEWORK #2, 10-14

1. Let x, y be generators of two copies of $\mathbb{Z}_2$ and let G be the free product of these copies. By definition of this product, the elements of G are exactly the powers $(xy)^n, (yx)^n = (xy)^{-n}$ for n a nonnegative integer and the products $x(xy)^n$ for n an arbitrary integer. It follows at once that xy has infinite order in G, x has order 2, and the conjugate of xy by x is $yx = (xy)^{-1}$. These properties are the ones defining the infinite dihedral group D_∞, whence $G \cong D_\infty$, as required.

2. (a) Begin by noting that the matrix $C = B'(A')^2$, so does indeed lie in the subgroup G of $SL_2(\mathbb{Z})$ generated by A' and B'. Multiplying a matrix M in $SL_2(\mathbb{Z})$ on the left by C^k amounts to adding k times the second row of M to its first row; multiplying M on the left by B' interchanges its two rows and then replaces the first row by its negative. Now one step of the Euclidean algorithm, applied to a pair (a, b) of integers not both 0,, replaces whichever of a, b has the larger absolute value by its remainder on division by the other, while leaving the other integer unchanged. Iterating this, we replace the original pair (a, b) by $(c, 0)$, where c is (say the positive) greatest common divisor of a and b. Applying this algorithm to the entries a, b in the first column of M and changing signs as necessary, we can replace this column by the one with entries $(1, 0)$, while the determinant of M is still 1. Then the entries of the second column of M must be $k, 1$ for some integer k, whence M is now the k-th power C^k of C. Hence G is all of $SL_2(\mathbb{Z})$, as claimed.

(b) It is immediate (as claimed in the problem statement) that the images A, B of A', B' in $PSL_2(\mathbb{Z})$ have orders 3 and 2, respectively. The linear fractional transformations T_1, T_2, and T_3 corresponding respectively to A, A^2, and B send z respectively to $(-z-1)/z = -1-(1/z), 1/(-z-1), -1/z$, whence indeed T_1 sends positive irrational numbers to negative ones less than $-1, T_2$ sends positive irrationals to negative ones greater than -1, and T_3 sends negative irrationals to positive ones. Now let $w_1 \ldots, w_k$ be a word of odd length whose letters are alternately A or A^2 and B. Conjugating it by B if necessary we may assume that it starts and ends with B. The corresponding product of T_1, T_2, T_3 then sends negative irrationals to positive ones, so cannot be the identity transformation. Similarly, if instead $w_1 \ldots w_k$ has even length k but is nonempty, then by conjugation we may assume that it starts with B and ends with A or A^2. Then the corresponding product of T_1, T_2, T_3 sends negative irrationals to negative irrationals less than -1 (if $w_k = A$) or to negative irrationals greater than -1 (if $w_k = A^2$), so cannot be the identity transformation in either case. We conclude that $PSL_2(\mathbb{Z})$ is the free product of its cyclic subgroups of orders 3,2 generated by A, B, respectively, as desired.

3. Observe first that $C^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = (BA^2)^2$ while similarly $(C^t)^2 = (BA)^2$. Examining products of powers of $(BA^2)^2$ and $(BA)^2$, we see that any such nonempty product reduces to a nonempty product of terms alternating between A or A^2 and B, which is not the

identity by the previous problem. Hence the subgroup of $PSL_2(\mathbb{Z})$ generated by C^2 and $(C^t)^2$ is freely generated by these elements, both of them having infinite order. Thus this subgroup is free on two generators, as desired.

4. Given the free group F_2 on two generators x, y it is immediate that the only possibilities (up to equivalence) for a Schreier transversal of a subgroup S of index 2 are $\{1, x\}$ and $\{1, y\}$. In the first case the element y of F_2 lies either in the identity coset of S or the coset of x; if the latter holds the coset of yx must be the identity coset, since S must be normal. Applying the recipe in class for the free generators of a subgroup of a free group, we get just three possibilities for these generators, namely $\{x, y^2, yxy^{-1}\}, \{y, x^2, xyx^{-1}\}$, or $\{x^2, yx^{-1}, xy\}$ (note that there is some latitude in the choice of generators in all three cases).

5. The easiest example of a subgroup of F_2 that is free on infinitely many generators (and thus necessarily of infinite index) is the normal subgroup generated by x. Here a Schreier transversal consists of all the powers of the other variable y and we get $\{y^i x y^{-i} : i \in \mathbb{Z}\}$ as a set of free generators of this subgroup. We could also take the set of such elements $y^i x y^{-i}$ with i running through the nonnegative integers only as generators of a different free subgroup of F_2.