SOLUTIONS TO HOMEWORK #5, DUE 11-4

1. (a) Let I be an ideal of R. If I=0, then there is only the 0 map from I to a K-vector space V, which extends to 0 on R, so assume that $i \neq 0$ and let $i \in I$. Then any R-module map f from I to a K-vector space V sends i to iv fro some $v \in V$, and if $i, j \in I$ are sent to $iv, jw \in V$, then looking at the image of ij we see that $v = w$. Hence there is a fixed $v \in V$ with $f(x) = xv$ for all $x \in I$, and f extends to the map from R to V sending r to rv. By Baer's Criterion, V is in injective over R.

(b) Look at the ideal $I = (x, y)$ generated by x and y in R and let $f : I \to K'$ send a combination $xp + yq$ to the image of q in K', for $p, q \in R$. As $xp = yq$ if and only if there is a polynomial r with $p = yr, q = xr$ (by unique factorization in R, it follows that f is well defined. If f extends to all of R, then $f(1)$ would have to be the image of $(xp + 1)/y$ in K' for some $p \in R$; but then $f(x) = x(xp + 1)/y \neq 0$ in K', a contradiction, since y cannot divide either x or $xp + 1$ for any p.

2. For the first part, look at the set of proper two-sided ideals; this is partially ordered by inclusion and the union of any chain of proper ideals is still proper, as each ideal in the chain excludes 1 and so the union does also. Hence there is a maximal proper two-sided ideal. The argument for left ideals is the same, as a proper left ideal must also exclude 1.

3. Letting f be an element of $D = \hom_R(S, S)$, we see that the kernel and image of f are both submodules of S, whence both are either all of S or 0. Hence either $f = 0$ or f is both one-to-one and onto and admits a two-sided inverse f^{-1}, which also lies in D, and D is a division ring.

4. First look at the left ideals of R. We know there is a maximal proper left ideal I, which admits a left ideal complement in R by projectivity; this complement must be simple as a left R-module, by maximality of I. Hence R has at least one (nonzero) minimal left ideal. Now look at the set of all collections $\{L_\alpha : \alpha \in A\}$ of left ideals in R such that the sum $\sum L_\alpha$ is direct. Such collections are partially ordered by inclusion and the union of chain of such collections is another one, so there is a maximal such collection. The sum of the ideals in it, if proper, lies in a maximal left ideal, which has a minimal complement as above; but then this ideal could be added to the maximal collection, a contradiction. Hence the sum is all of R. But the element $1 \in R$ is the sum of finitely many elements, each from one ideal in the collection, whence the finitely many ideals so involved already have direct sum R, and R is the direct sum of finitely many minimal left ideals.

It follows at once that R satisfies the descending chain condition on left or two-sided ideals: given the direct sum $R = \oplus i = 1^n L_i$, any infinitely strictly descending chain of left ideals would give rise to such a chain either in L_1 or $R/L_1 \cong \oplus_{i=2}^n L_i$, which is impossible by induction. It follows that any nonempty set of left ideals or two-sided ideals in R has a minimal element.

Thus R has at least one minimal two-sided ideal I, which has a left ideal complement J. Writing 1 as $e+f$ where $e \in I, f \in J$, we see that the left R-submodules $Re, Rf = R(1-e)$ of I, J already have sum R, whence $I = Re, J = R(1-e)$. Then $eR(1-e) \subset I \cap J = 0$, whence $R(1-e) \subset (1-e)R$ (since R is also the direct sum of eR and $(1-e)R$ and $e(ex) = ex$ for all $x \in R$). It follows that $(1-e)R$ is a two-sided ideal of R, which does not contain e (since $ey = 0$ for all $y \in (1-e)R$, and the intersection $(1-e)R \cap Re = 0$ by minimality of Re. But the sum of Re and $(1-e)R$, as a right ideal, must be all of R, whence finally $J = (1-e)R$ is a two-sided ideal complementary to I. Looking at two-sided R-subideals of J, we find another minimal one, which again admits a two-sided complement, and so on; in this way we get a direct sum $R_1 \oplus R_2 \oplus \cdots$ of minimal two-sided ideals of R. This sum must terminate after finitely many steps with a decomposition of all of R, by the descending chain condition, so at least we get the decomposition claimed. Writing 1 as $\sum e_i$ with $e_i \in R_i$, we check immediately by multiplication that $\sum_i e_i = 1, e_i e_j = 0$ if $i \neq j, e_i^2 = e_i$, and at last we are done.

5. As in the first part of the last problem, write each R_i as the direct sum of finitely many simple left ideals L_{ij}. Given two such ideals, say L_{i1}, L_{i2}, note first that the annihilator $\{x \in R_i : xLi, 2 = 0\}$ of L_{i2} is a proper two-sided ideal, so must be 0, and there is $x \in L_i 2$ with $L_{i1}x \neq 0$; but then $L_{i1}x$ is a nonzero submodule of L_{i2}, which must be all of L_{i2}. Similarly, $\{y \in L_{i1} : yx = 0\}$ is a submodule of L_{i1}, which must be 0, so we conclude that $Li1 \cong L_{i2}$, as desired.