SOLUTIONS TO HOMEWORK #4, DUE 10/28

1. First suppose that the matrix M is the companion matrix $C(p)$ attached to a single monic polynomial p. The minimal polynomial of $C(p)$ is p itself, whence the same is true of its transpose $C(p^t)$, since a polynomial q vanishes on a matrix M if and only if it vanishes on M^t. But a matrix in (the invariant factor version of) rational canonical form, having blocks the companion matrices of $p_1, \ldots, p_m$ with $p_1|p_2|\cdots p_m$, has minimal polynomial p_m, whence the degree of this polynomial equals the size of the matrix if and only if there is just one block. Hence $C(p)$ is the only possible rational canonical form for $C(p)^t$, and $C(p)^t$ is similar to $C(p)$, as desired. Now a matrix in block diagonal form with blocks $B_1, \ldots, B_m$ similar respectively to square matrices $C_1, \ldots C_m$, is easily seen to be similar to the block diagonal matrix with blocks $C_1, \ldots, C_m$, so the desired result now follows from the rational canonical form.

2. This follows at once from the rational canonical form in its invariant factor version: since two polynomials p_1, p_2 in $K[x]$ are such that $p_1|p_2$ in $K[x]$ if and only if $p_1|p_2$ in $L[x]$ for L a field containing K, it follows that the only possible rational canonical form over L for a matrix over K is the same as this form over K.

3. A projective module over any ring is a direct summand of a free module; over a PID R, any free module is torsion-free, since R is an integral domain, so a finitely generated projective R-module cannot involve any proper quotients $R/(q)$ and must be a finite direct sum of copies of R. Thus the finitely generated projective R-modules are exactly the free ones R^m of finite rank.

4. If M is free with basis $b_1, \ldots, b_n$, then I claim that $\bigwedge^k M$ is also free, with basis $b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_k}$, where the i_j range over all indices between 1 and n with $i_1 < i_2 < \ldots < i_k$; in particular, the rank of this module is

$$\binom{n}{k} = n!/(k!(n-k)!)$$

To see this, it is enough to show (as we did in class for the full tensor power $T^k M$) that an alternating k-linear function f from $M \times \cdots M$ to another R-module N is completely determined by the images $f(b_{i_1}, \ldots, b_{i_k})$ of tuples of basis vectors with indices as above, and these images are arbitrary (so that any choice of them gives rise to a unique alternating k-linear map). It is clear that $f(b_{i_1}, \ldots, b_{i_k})$ is determined for *any* k-tuple of indices i_j by the values of $f(b_{i_1}, \ldots, b_{i_k})$ for $i_1 < \ldots < i_k$, since then $f(b_{i_{\sigma(1)}}, \ldots, b_{i_{\sigma(k)}})$ equals the sign of σ times $f(b_{i_1}, \dots, b_{i_k})$ for any permutation σ of $1, \dots, k$, while $f(b_{i_1}, \dots, b_{i_k}) = 0$ whenever two indices i_j are equal. So it remains to show that any choice of $f(b_{i_1}, \dots, b_{i_k})$ for all indices with $i_1 \dots < i_k$ gives rise to an alternating multilinear f defined on all of M^k. This follows since a formula for f is given by $f(m_1, \dots, m_k) = \sum_{i_1 < \dots i_k} M_{i_1,\dots i_k} f(b_{i_1}, \dots b_{i_k})$, where the matrix $M_{i_1,\dots i_k}$ has its jth column consisting of the coefficients of $b_{i_1}, \dots b_{i_k}$ when m_j is written as a combination of $b_1, \dots b_n$. That such an f is alternating and k-linear follows from standard properties of determinants (over commutative rings).

5. Write the $\mathbb{Z}$-module M as F/N with F a free $\mathbb{Z}$-module, and let F' the free $\mathbb{Q}$-module (or vector space over $\mathbb{Q}$) on the same basis as F. Then F'/N contains M and is divisible and thus injective over $\mathbb{Z}$, as desired.