HOMEWORK #5, DUE 11/4

MATH 504A

1. (a) Let R be an integral domain with field of fractions K. Show that any vector space over K is injective as an R-module.

(b) Let $R = k[x, y]$, the ring of polynomials in two variables x, y over a field k, K the field of fractions of R. Show that the quotient $K' = K/Rx$ of K by the ideal generated by x in R is not injective as an R-module, by exhibiting a nonprincipal ideal I of R and an R-module map from I to K' that does not extend to R.

2. In the remaining problems R is a non necessarily commutative ring (always with 1). Using Zorn's Lemma, show that R has a maximal (proper) two-sided ideal and a maximal left ideal.

3. Let S be a simple left R-module (so that S has no submodules apart from itself and 0). Show that the ring $\mathrm{Hom}_R(S, S)$ of R-homomorphisms from S to itself is a division ring (obeying all axioms of a field except commutativity of multiplication).

4. Assume now that every left R-module is projective. Use Zorn's Lemma to show that R is the direct sum of simple two-sided ideals R_i, each generated as a left R-module by a single element e_i, and that there are only finitely many such ideals $R_1, \ldots, R_n$. (You may assume that every two-sided ideal R contains a minimal one) Show also that we may choose the e_i so that $\sum_i e_i = 1, e_i^2 = e_i, e_i e_j = 0$ if $i \neq j$.

5. Continuing with the setting of the last problem, show that each R_i is in turn the direct sum of finitely many simple left ideals of R_i and that any two such ideals are isomorphic as left modules.