HOMEWORK #3, DUE 10/21

MATH 504A

1. Let M be an $n \times n$ matrix over a commutative ring R with $\det M = 0$. Show that there is a nonzero $v \in R^n$ with $Mv = 0$, by first letting k be the largest positive integer (if any) with some $k \times k$ submatrix of M having nonzero determinant, and using determinants of suitable $k \times k$ submatrices of M as the coordinates of v. Deduce that no R-module map from R^n to R^m can be injective if $n > m$.

2. Show that the ring R of linear transformations from the direct sum $\mathbb{R}^\infty$ of countably many copies of the real numbers $\mathbb{R}$ to itself is such that $R \cong R \oplus R$ as an R-module, by dividing a basis for the domain of any such transformation into two countably infinite subsets.

3. Show that the direct *product* $M = \mathbb{Z}^\omega$ of countably many copies of $\mathbb{Z}$, consisting by definition of all sequences $(z_1, z_2, \dots)$ with the $z_i \in \mathbb{Z}$ but no other restriction is not a free $\mathbb{Z}$-module, as follows. First note that M is uncountable, while the direct sum $N = \mathbb{Z}^\infty$ consisting of all sequences with all but finitely many z_i equal to 0 is countable. If M had a basis B over $\mathbb{Z}$, then some countable subset of B, say B', would span N. Let M' be the quotient of M by the span of B'; then M' would be free with basis the images in it of the elements in B but not B'. The span of B' is countable, so at least one of the uncountably many elements $(\pm 1!, \pm 2!, \dots)$ has nonzero image v in M'. Show that for any nonzero integer i there is $v_i \in M'$ with $iv_i = v$; but no nonzero element of a free $\mathbb{Z}$-module has this property.

4. Classify the finitely generated $\mathbb{Z}$-submodules of $\mathbb{Q}$ and show in particular that the *subring* of $\mathbb{Q}$ generated by $1/2$ is not finitely generated as a $\mathbb{Z}$-module.

5. Show that the tensor product (over $\mathbb{Z}$) of the $\mathbb{Z}$-modules $\mathbb{Z}_m$ and $\mathbb{Z}_n$ is 0 whenever m, n are relatively prime integers.