HW #8, due 11-22

Math 504A

1. Let π be an irreducible representation of a finite group G with character χ. Show that $\chi(g) = \chi(1)$ if and only if g lies in the kernel of the representation π, and that if $g \neq 1$ then there is an irreducible character χ with $\chi(g) \neq \chi(1)$.

2. Let G be a finite group and H a subgroup such that $g^{-1}Hg \cap H = 1$ for $g \notin H$. Show that the set N of all elements in G not conjugate to any element of H, together with 1, is a normal subgroup of G, using the following steps. First show that if c_H is a class function on H with $c_H(1) = 0$, then the induced function $c_G = \mathrm{Ind}_H^G c_H$ has the same square length as c_H. Next let χ_H be an irreducible character of H of degree $d = \chi(1)$ and let c_χ be the function induced from $\chi_H - d$ to G. Use Frobenius reciprocity to show that c_χ takes the form $\chi_G - d = \chi_G - \chi_G(1)$ for some irreducible character χ_G of G. Letting χ_H run through the irreducible characters of H, show that N is the set of common zeros of the c_χ in G and thus the intersection of the kernels of the representations π_G with character χ_G.

3. (20 points) Let G be a nonabelian group of order p^3, p a prime. Work out the character table of G, as follows. Show that its center Z has order p and that G/Z is the product of two cyclic groups of order p. Determine the number of irreducible one-dimensional representations and work out the degrees of all the other representations. Describe the conjugacy classes in G and finally work out the value of every irreducible character on every conjugacy class. In particular, show that any two nonabelian groups of this order have isomorphic character tables, though they need not be isomorphic.

4. For any partition λ of n, the construction of the Specht module S^λ makes sense over any basefield k. Show that if k has characteristic 3 and $\lambda = (2, 1)$, then $V = S^\lambda$ is *not* irreducible over S_3, by working out the actions of a 3-cycle and a transposition explicitly on the basis of V parametrized by standard tableaux of shape λ.