HW #7, due 11/15

Math 504A

1. Let R be any ring and M an irreducible left R-module (so that M has no nonzero proper R-submodule). Show that the ring $\hom_R(M, M)$ of R-module homomorphisms from M to itself is a division ring; that is, it satisfies all axioms of a field except for commutativity of multiplication.

2. Recall that the dihedral group D_8 of order 8 is generated by two elements x, y such that $x^4 = y^2 = 1, yxy = yxy^{-1} = x^{-1}$. This group is the group of symmetries of a square. Use this fact to choose vertices of a suitable square and write down an irreducible two-dimensional real representation π of D_8, computing the 2×2 matrices $\pi(x)$ and $\pi(y)$ explicitly.

3. Recall that the quaternion group Q_8, also of order 8 and generated by two elements x, y, this time such that $x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1}$, is *not* isomorphic to any subgroup of the group $G = GL(2, \mathbb{R})$ of 2×2 invertible real matrices. (Assume contrarily that π is an isomorphism from Q_8 into a subgroup of G. Show first that $\pi(x^2) = -I$, the negative of the identity matrix. Replacing π by a conjugate, you may assume that $\pi(x)$ is in rational canonical form. Then deduce a contradiction).

4. Construct character tables for the dihedral and quaternion groups of order 8, making sure to identify and label the conjugacy classes of these groups.

5. If V, W are two representations of the same group G over the same field k, then their tensor product $V \otimes_k W$ becomes a representation of G via the recipe $g(v \otimes w) = gv \otimes gw$ for $g \in G, v \in V, w \in W$. Show that the character $\chi_{V \otimes W}$ of $V \otimes W$ is the product of the characters χ_V, χ_W of V, W, respectively.

Also read sections 18.1-18.3, 19.1, and start 19.2.