HW #6, due 11-8

Math 504A

1. Show that the characteristic polynomial of the companion matrix $C(f)$ is f (up to sign) for any monic polynomial f.

2. (20 points) Let V be the infinite-dimensional module of the Klein four-group $K = C_2 \times C_2$ over the field F_2 with two elements given in class, so that V has a basis $\{v_i : i \in \mathbb{Z}\} \cup \{w_i : i \in \mathbb{Z}\}$ over F_2 and the commuting generators x, y of K are such that $xw_i = yw_i = w_i, xv_i = v_i + w_{i+1}, yv_i = v_i + w_i$. Show that V is indecomposable. (Assume for a contradiction that $V = V_1 \oplus V_2$ for some nonzero submodules V_1, V_2. Show that the projections P_1, P_2 of V_1, V_2 to the span S of the v_i intersect trivially and that S is their direct sum. Then $(x-1)P_1 + (x-1)P_2 = (y-1)P_1 + (y-1)P_2 = T$, the span of the w_i, forcing $(x-1)P_1 = (y-1)P_1, (x-1)P_2 = (y-1)P_2$. Whenever a particular combination $\sum a_i v_i$ lies in P_1, this last fact forces a large number of other such combinations to lie in P_1, and similarly for P_2. Deduce a contradiction unless one of P_1, P_2, say P_1, is all of S. Then show that $V_1 = V$.)

4. (20 points) Let $q = p^n$ be a power of a prime p. A *deBruijn sequence for q* is a sequence $s = a_0 \ldots a_{q-1}$ of q symbols, each an element of the field F_p of order p, such that every sequence $b_1 \ldots b_n$ of n symbols, each an element of F_p, occurs exactly once as a consecutive subsequence $a_i \ldots a_{i+n-1}$ of s for $i \in [0, q-1]$; here addition of subscripts takes place modulo q, so that the sequence wraps around. Show that a deBruijn sequence exists by the following steps. Start with the field F_q of order q and show that it has a nonzero element x of multiplicative order $q-1$. Let f be the minimal polynomial of x over F_p, so that f has degree n. Then the matrix of multiplication by x on F_q, regarded as an n-dimensional vector space over F_p, is the transpose $M = C(f)^t$ of $C(f)$ with respect to some basis, by a result in class. Now define the sequence s by decreeing first that $v = (a_0, \ldots, a_{n-1}) = (0, \ldots, 0, 1)$ and then that the vector $(a_i, \ldots, a_{i+n-1}) = (M^i v^t)^t$, the transpose of the column vector $M^i v^t$, for $0 \le i \le q-n$. Show that this makes sense and yields a deBruijn sequence $a_0 \ldots a_{q-1}$, making sure to check also that $00\ldots0$ is a consecutive subsequence of s.