HW #1, due 10-4

Math 504A

1. (20 points) Let G be a finite group of order $p^k m$, where p is prime and does not divide m. Let ℓ an integer between 1 and k.

(a) Let G act on the set of its subsets of size p^ℓ by left translation. Show that any such subset S is the union of right cosets of its stabilizer and deduce that this stabilizer has order p^r for some $r \leq \ell$. The orbit of S thus has size a multiple of $p^{k-\ell+1}m$ if $r < \ell$ and size exactly $p^{k-\ell}m$ if $r = \ell$. Deduce that the congruence class mod p of the number $n_{p,\ell}$ of subgroups of G of order p^ℓ depends only on the order of G, not on its structure; by looking at a cyclic group of order $p^k m$, deduce that $n_{p,\ell} \equiv 1 \mod p$. In particular $n_{p,\ell} \neq 0$.

(b) Thus G has at least one subgroup of order p^k; such a subgroup is called a *p-Sylow subgroup*. Let $r \leq k$ and consider the action of a subgroup G_r of G of order p^r on the left cosets of a p-Sylow subgroup P by left translation. Show that some orbit has size one and deduce that G_r lies in some conjugate of P.

(c) In particular, any two p-Sylow subgroups of G are conjugate. Deduce from this that the number $n_{p,k}$ of p-Sylow subgroups divides m (in addition to being congruent to 1 mod p).

2. Show that every group of order 56 has a normal subgroup, by showing that either the 2-Sylow or the 7-Sylow subgroup of such a group must be (unique and) normal; more specifically, if a 7-Sylow subgroup is not normal, then there are too few elements of G with order a power of 2 for it to have more than one 2-Sylow subgroup.

3. Let $q = p^k$ be a power of a prime p. Assume that there exists a field F_q with q elements (we will see later that this is always the case). Compute the order of the group $G = GL(n, F_q)$ of invertible n-by-n matrices over F_q, by counting how many choices there are for the first row of such a matrix, then counting how many choices there are for the second row, given the first one, and so on.

4. Now you know the largest power p^N dividing the order of G. By looking at triangular matrices in G, find a p-Sylow subgroup of it.

Also read Chapter 4 in the text (and review previous chapters as necessary).