FINAL EXAM SOLUTIONS–MATH 504A

1. Is it true that two representations π, π' of a finite group G are such that $\det \pi(g) = \det \pi'(g)$ for all $g \in G$, then π and π' are equivalent?. If so, prove it; if not, give a counterexample.

False; probably the easiest counterexample is the direct sum of two copies of the trivial representation and the trivial representation itself; in both cases the determinant is constantly equal to 1.

2. How many similarity classes are there of 5×5 matrices over $\mathbb{Q}$ whose minimal polynomial is $(x-1)^2(x-2)$ and whose characteristic polynomial is $(x-1)^3(x-2)^2$? Give a representative from each class.

. There must be two Jordan blocks with eigenvalue 1, one of size 2×2 and the other 1×1, and two blocks iwth eigenvalue 2, both of size 1×1. Hence there is just one similarity class of such matrices, represented by a block diagonal matrix with the given blocks.

3. Let $p \in \mathbb{Z}$ be prime. Compute the groups $\mathrm{Ext}^n_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all n, regarding $\mathbb{Z}_p$ as a module over itself in the obvious way.

$\mathbb{Z}_p$ is free over itself and thus projective, whence the Ext groups are 0 for $n \geq 1$, while $\mathrm{Ext}^0_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$.

4. Show that the *real* group algebra $\mathbb{R}G$ of a finite group G need *not* be a direct sum of matrix rings over $\mathbb{R}$.

Take $G = \mathbb{Z}_3$; as G is abelian, if $\mathbb{R}G$ were a sum of matrix rings, it would have to be a sum of copies of $\mathbb{R}$. But there is no element of order 3 in any such sum of copies.

5. Give a simple necessary and sufficient condition for a module over $\mathbb{Q}[x]$ to be injective.

By a result in class, such a module M is injective if and only if it is divisible, so that $M = qM$ for all nonzero $q \in \mathbb{Q}[x]$.

6. Let P be a p-group for some prime p acting on a set S whose cardinality is not a multiple of p. Show that there is $s \in S$ fixed by every element of P.

By the Orbit Formula, every orbit has size a power of p; since the orbits are disjoint, the only way for the cardinality of S not to vbe a multiple of p is if an orbit of size 1, and so a fixed point, occurs.

7. Classify the abelian groups of order 16.

There correspond to the partitions of 4, since $16 = 2^4$ is a power of the prime 2. Hence there are five such groups: $\mathbb{Z}_{16}, \mathbb{Z}_8 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $(\mathbb{Z}_2)^4$.

8. Let G be a finite group. Show that every module over the group algebra $\mathbb{C}G$ is projective.

Every such module M is a direct sum of irreducible modules, each of which is a direct summand of the free module $\mathbb{C}G$, so M is projective.