Lecture 9-27: Group actions and products

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I will continue with group actions, beginning with one of the most important and interesting concrete examples of them, to which I will return on a number of occasions. I will then take amore abstract point of view, looking at groups acting on other groups by automorphisms.

First let G be a subgroup of the group $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices over $\mathbb{R}$. Then for any $g \in G$ and $v \in \mathbb{R}^n$, regarded as a column vector, one has the matrix product gv, another column vector in $\mathbb{R}^n$; it is clear that the rule $g \cdot v = gv$ defines an action of G on $\mathbb{R}^n$. In particular, given a subset S of $\mathbb{R}^n$, the set of all g in $GL_n(\mathbb{R})$ (or in a suitable subgroup H) preserving S is a subgroup of $GL_n(\mathbb{R})$ acting on S (by symmetries). In particular, taking $n = 3$ and $H = SO_3(\mathbb{R})$, the group of orientation-preserving (and length- and angle-preserving) symmetries of $\mathbb{R}^3$, one finds that the set of such symmetries preserving S, which we may call its symmetry group, acts on S in a natural way. We can also characterize $SO_3(\mathbb{R})$ as the set of all $M \in GL_3(\mathbb{R})$ such that $\det M = 1$ and $M^t = M^{-1}$, where M^t denotes the transpose of M.

I now consider some very particular subsets of $\mathbb{R}^3$, namely the Platonic solids (or regular polyhedra). I highly recommend consulting the Wikipedia page for these, which features some very cool animations that help a lot to visualize them. These consist of the tetrahedron, having four triangular faces, six edges, and four vertices; the cube, having six square faces, twelve edges, and eight vertices; the octahedron, having eight triangular faces, twelve edges, and six vertices; the dodecahedron, having twelve pentagonal faces, thirty edges, and twenty vertices; and the icosahedron, having twenty triangular faces, thirty edges, and twelve vertices. There are only three distinct symmetry groups of these objects, having orders 12, 24, and 60.

In more detail, the tetrahedron has symmetry group (isomorphic to) A_4, the group of even permutations of its four vertices; the cube and octahedron have symmetry group S_4, consisting of all permutations of its four pairs of opposite vertices (for the cube) or opposite faces (for the octahedron). The last two cases of the dodecahedron and icosahedron are the most subtle of all: the symmetry group of both of these is A_5, consisting of all even permutations of five objects, these being the five cubes that one can inscribe in the dodecahedron (see the Wikipedia page on the dodecahedron).

The reason that the cube and octahedron have the same symmetry group is that any cube has an octahedron inscribed in it and vice versa: if you take the center of each face of a cube and join the centers of two faces whenever the faces have a common edge, then you get an octahedron, and vice versa. Similarly, the dodecahedron and icosahedron have the same symmetry group, for the same reason.

That the symmetry group of each solid has the claimed order follows at once from the Orbit Formula. In each case, the symmetry group acts transitively on the vertices, edges, and faces. For example, the tetrahedron is such that each face has stabilizer of order 3, consisting of the three rotations (by 0, 120, and 240 degrees) of it about its center, so its symmetry group has order 12. Each face of the cube similarly has exactly four rotations stabilizing it, so the symmetry group of the cube has order 24. The dodecahedron has five rotations stabilizing each face, so a symmetry group of order 60. We will see below that A_4 is the only subgroup of S_4 of order 12 and that A_5 is the only subgroup of S_5 of order 60.

If one allows symmetries preserving lengths and angles but not necessarily orientations, corresponding to matrices $M \in GL_3(\mathbb{R})$ with $M^t = M^{-1}$ but $\det M$ allowed to be ± 1, then the symmetry group of the tetrahedron enlarges to S_4; all permutations of its four vertices are now allowed. The cube and dodecahedron behave differently. Each vertex has an opposite vertex, so the transformation -1 (the negative of the identity) is now a symmetry. Accordingly the full symmetry groups of the cube/octahedron and dodecahedron/icosahedron are the direct product $S_4 \times \mathbb{Z}_2$ and $A_5 \times \mathbb{Z}_2$, respectively, where $\mathbb{Z}_2$ denotes the cyclic group of order 2.

Turning now to a more general setting, and moving on to Chapter 5 in the text, let H be a group acting on another group G by automorphisms, so that the associated homomorphism from H to $\mathrm{Perm}(G)$ in fact sends H into the automorphism group $\mathrm{Aut}\ G$, we can construct a new group, called the semidirect product of G and H and denoted $G \rtimes H$, as follows. Start with the Cartesian product $G \times H$ and define a product on it via $(g_1, h_1)(g_2, h_2) = (g_1(h_1 \cdot g_2), h_1 h_2)$. It is easy to check that the group axioms are satisfied. Note that multiplication in the second coordinate behaves exactly as it would for the direct product $G \times H$, while in the first coordinate the term g_2 is "twisted" by the action of h_1; this is why the term "semidirect" is used. If the action of H on G is trivial, so that every $h \in G$ fixes every $g \in G$, then $G \rtimes H$ reduces to the direct product $G \times H$. See Theorem 10 on p. 176.

A group K not given to us as the semidirect product of subgroups G and H might be isomorphic to such a semidirect product anyway. To decide when this happens, recall first that if K has normal subgroups G, H such that $K = GH$ and $G \cap H = 1$, then K is isomorphic to the direct product $G \times H$. To see this one argues first that if g_1, g_2, h_1, h_2 are such that $g_1 h_1 = g_2 h_2$ and $g_1, g_2 \in G, .h_1, h_2 \in H$, then $g_1 g_2^{-1} = h_2 h_1^{-1} \in G \cap H = 1$, whence $g_1 = g_2, h_1 = h_2$. Next one observes that for $g \in G, h \in H$ the commutator $ghg^{-1}g^{-1} \in G \cap H = 1$, so that $gh = hg$. The isomorphism $K \cong G \times H$ follows at once.

Now suppose that all of the above hypotheses on K, G, and H hold, *except* that we do not assume that H is normal in K. We still have every $k \in K$ equal to the product gh for a unique $g \in G, h \in H$ as above, but now a typical commutator $ghg^{-1}h^{-1}$ lies in G but not necessarily in H, so that it need not equal 1. We still have an action of H on G by conjugation, so that $h \cdot g = hgh^{-1}$ and the map sending g to hgh^{-1} (for fixed $h \in H$) is an automorphism of G. In this case it is easy to check that K is isomorphic to the product $G \rtimes H$ relative to the above action of H on G. See the discussion on p. 175.

As an example, let K be the symmetric group S_n of all permutations of $\{1, \ldots, n\}$ for some $n \geq 2$ and let G, H be the alternating group A_n and the cyclic subgroup generated by the transposition t interchanging 1 and 2, respectively. Then K is the semidirect product of G and H, with H acting on G by conjugation. As another finite example, let R_n be the group of rotational symmetries of a regular n-gon P_n in the plane (which is cyclic of order n) and S the cyclic subgroup generated by any reflectional symmetry of P_n (which is cyclic of order 2). Then the dihedral group D_n, the full symmetry group of P_n, is the semidirect product of R_n and S. Further finite examples are given on pages 178 and 179.

Note that even if K has a nontrivial normal subgroup G it is not necessarily isomorphic even to the semidirect product of G and any other subgroup. For example, the quaternion group Q of order 8 has three cyclic subgroups of order 4, all of them normal; it also has a unique cyclic subgroup H of order 2. But each of the subgroups of order 4 contains the subgroup of order 2, so that Q is not the semidirect product of any normal subgroup and H. We say that Q is a nonsplit extension of its quotient $\mathbb{Z}_2$ by any of its cyclic subgroups of order 4. (By contrast, as noted above, the dihedral group D_4 of order 8 is the semidirect product of its subgroup R_4 of rotations and any of its subgroups generated a single reflection). It turns out that any nonabelian group of order p^3, where p is an odd prime, is a semidirect product (see p. 179).

The group A_4 is very special among alternating groups in that it is the semidirect product of the Klein 4-group K (consisting of the identity and all three products of two disjoint transpositions) and the cyclic subgroup generated by any 3-cycle. Likewise S_4 is the semidirect product of K and the symmetric group S_3. But the alternating group A_n is simple (has no nontrivial proper normal subgroups) for $n \geq 5$, as we will see below, so that it cannot be realized as a nontrivial semidirect product.

Recall the cycle notation for permutations (pp. 29,30): the permutation $\pi \in S_n$ sending the index i_1 to i_2, i_2 to $i_3, \ldots, i_{m-1}$ to i_m, i_m back to i_1, then j_1 to j_2, j_2 to j_3, and so on, is denoted $(i_1\, i_2 \ldots\, i_m)(j_1\, j_2 \ldots) \ldots$ Then the conjugate $\sigma\pi\sigma^{-1}$ of π by $\sigma \in S_n$ sends $\sigma(i_1)$ to $\sigma(i_2)$, and so on, so that its cycles have the same lengths as those in π. Thus there are five conjugacy classes in S_4, represented by a 4-cycle, a 3-cycle, the product of two disjoint 2-cycles, a single 2-cycle, and the identity, of respective sizes $6, 8, 3, 6$, and 1.

A subgroup of S_4 of order 12 and index 2 must be normal, so is a union of conjugacy classes closed under multiplication, from which one easily checks (as claimed above) that the only such subgroup is A_4. Similarly, S_5 has seven conjugacy classes, represented by a 5-cycle, a 4-cycle, the product of disjoint 3- and 2-cycles, a 3-cycle, the product of two disjoint 2-cycles, a single 2-cycle, and the identity, of respective sizes $24, 30, 20, 20, 15, 10$, and 1. A subgroup of order 60 is again normal and closed under multiplication, so must be A_5. Similarly, one can show that A_5 is simple: its conjugacy classes have sizes 12,12,20,15, and 1.