# Lecture 12-2: Review

December 2, 2024

This week will be devoted to review, mostly in the chronological order that topics were originally presented.

In group theory, I concentrated on group actions on sets, the central result being the Orbit Formula that given a finite group $G$ acting on a set $S$ and $s \in S$, the product of the orders of the stabilizer $G^s$ of $s$ in $G$ and the orbit $G.s$ through $s$ equals the order of $G$. In particular, using the conjugation action of a group on itself, I showed that any group $G$ of order $p^n$ for a prime $p$ admits normal subgroups of all possible orders $p^m$ for $m \leq n$ and a nontrivial center $Z$. Moreover, a group of order $p^2$ is abelian.

The structure of arbitrary groups is related to that of $p$-groups via Sylow's Theorems, which state that

1. Any group of order $p^a r$ for $p$ a prime not dividing $r$ has a subgroup of order $p^a$.
2. Any two such subgroups are conjugate.
3. The number $n_p$ of such subgroups satisfies $n_p | r$ and $n_p \equiv 1 \mod p$.

In particular, if $p, q$ are primes with $p < q$, then the $q$-Sylow subgroup of any group $G$ of order $pq$ is (unique and) normal; the $p$-Sylow subgroup is also normal and in fact $G$ is cyclic if $q \not\equiv 1 \mod p$.

Turning now to left modules $M$ over rings, recall that $M$ is projective if the covariant functor $\hom_R(M, -)$ is exact, or equivalently if given any surjective map $\pi : N \to P$ of $R$-modules and a homomorphism $f : M \to P$ there is a homomorphism $g : M \to N$ with $f = \pi g$. The functor $\hom_R(M, -)$ is always left exact, but not in general right exact. $M$ is projective if and only if it is a direct summand of a free $R$-module $F$ (so that $F$ is isomorphic to a direct sum of copies of $R$). Any projective module over $R$ is free if $R$ is a PID, but not in general, even if $R$ is commutative. Any module is a homomorphic image of a free (and thus projective) module.

In a formally very similar way, an $R$-module $M$ is injective if the contravariant functor $\hom_R(-.M)$ is exact, or equivalently if any $R$ module map from $A$ to $M$ always extends to a map from $B$ to $M$ whenever $A$ is a submodule of $B$. The functor $\hom(-, M)$, like $\hom(M, -)$ is always left exact. There is no simple characterization of injective modules in general. Over a PID, $M$ is injective if and only if it is divisible, so that $M = rM$ for for all $r \neq 0$ in $R$. This is *not* the same as saying that $M$ is a module over the quotient field $K$ of $R$, though certainly any such module is injective over $R$. Any module is a submodule of an injective one.

The tensor product $M \otimes_R N$ of two modules $M, N$ over a commutative ring $R$ may be defined as the quotient of the free module on the Cartesian product $M \times N$ by the submodule generated by $(m_1 + m_2, n) - (m_1, n) - (m_2, n), (m, n_1 + n_2) - (m, n_1) - (m, n_2), r(m, n) - (rm, n)$ and $r(m, n) - (m, rn)$ for $m, m_i \in M, n, n_i \in N, r \in R$. If $M$ and $N$ are free of ranks $r$ and $s$ over $R$, then $M \otimes RN$ is free of rank $rs$ over $R$, but in general the size of $M \otimes_R N$ may be hard to predict. It is possible for $M \otimes_R N$ to be 0 even if $M$ and $N$ are nonzero, for example. Also the ring $R$ is crucial here: if the same $M, N$ are simultaneously modules over different rings $R, S$, then $M \otimes_R N$ may be very different from $M \otimes_S N$.

One can also tensor one-sided modules over noncommutative rings. More precisely, if $M$ and $N$ are respectively right and left modules over the same ring $R$, then the tensor product $M \otimes_R N$ makes sense, replacing the relations $r(m,n) - (rm,n), r(m,n) - (m,rn)$ above by the single relation $(mr,n) - (m,rn)$, This product has only the structure of an abelian group, unless $M$ also carries a left module structure for another ring $S$ commuting with the right $R$-module structure; in that case $M \otimes_R N$ is also a left $S$-module. This construction was used in class to define the module $W = \mathbb{C}G \otimes_{\mathbb{C}H} V$ induced from a module $V$ over a subgroup $H$ of a group $G$; then $W$ has a $G$-module structure.

Analogous to the functors $\hom_R(M, -)$ and $\hom_R(-, M)$ mentioned above from left $R$-modules to abelian groups (where $M$ is a fixed left $R$-module) one has the functor $M \otimes_R -$ from left $R$-modules to abelian groups, where now $M$ is a fixed right $R$-module. This time the functor is right but not left exact, in general; if it is (left) exact, then the module $M$ is called flat. Projective modules are flat, but not conversely; if for example $R$ is an integral domain, then any module over its quotient field $K$ is flat as an $R$-module.

A projective resolution of an $R$-module $M$ is an exact (possibly infinite) sequence $\cdots \to P_1 \to P_0 \to M$ of $R$-module maps with the $P_i$ projective over $R$; such a sequence exists for any $M$ and in fact the $P_i$ can be taken to be free over $R$. Similarly, one has an injective resolution $M \to I_0 \to I_1 \cdots$ of $M$, where the sequence is exact and the $I_i$ are injective. Given another $R$-module $N$ and a projective resolution $\{P_i\}$ of $M$, one can apply the functor $\hom_R(-, N)$ to each term, obtaining the cochain complex $\hom_R(P_0, N) \to \hom_R(P_1, N) \to \cdots$ The $i$th cohomology group of this complex, that is, the kernel of the map from $\hom_R(P_i, N)$ to $\hom_R(P_{i+1}, N)$ modulo the image of the map from $\hom_R(P_{i-1}, N)$ to $\hom_R(P_i, N)$, is defined to be the Ext group $\text{Ext}^n_R(M, N)$. The 0th such group $\text{Ext}^0_R(M, N)$ is just $\hom_R(M, N)$. The Ext groups do not depend on the choice of projective resolution.

The groups $\mathrm{Ext}_R^n(M, N)$ provide a measure of non-projectivity of $M$, since $M$ is projective if and only if $\mathrm{Ext}_R^n(M, N) = 0$ for all $R$modules $N$ and integers $n \geq 1$. It often happens that the Ext groups have a periodic structure if they do not vanish for sufficiently large $n$.

Next time I will review the classification of finitely generated modules over a PID and the applications of this to canonical forms of matrices. I will also start to review the representation theory of finite groups.