Lecture 11-6: Characters and character tables

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I continue with characters χ of irreducible representations π, calling them irreducible characters. I will show that it is indeed possible (in principle) to recover π from knowledge of χ alone. Throughout I assume for simplicity that the basefield k is the complex field, though the results can be formulated to apply to any algebraically closed field of characteristic not dividing the order n of the finite group G.

I begin with a few basic facts. Let $\pi : G \to GL(V)$ be a representation and χ its character. Note that the map sending $g \in G$ to $(\pi(g)^{-1})^t$ is also a representation, called the dual of V and denoted V^*.

Proposition 14, p. 872

- $\chi(g)$ is a sum of roots of 1 for all $g \in G$.
- $\chi(g^{-1}) = \overline{\chi(g)}$, the complex conjugate of $\chi(g)$.
- Given a direct sum $V = \oplus V_i$ of representations of G, the character of G on V is the sum of the characters of G on the V_i.
- The character χ' of V^* satisfies $\chi'(g) = \overline{\chi(g)}$.

Proof.

The first part follows since the trace of a matrix is the sum of its eigenvalues and the eigenvalues of any $\pi(g)$ are all roots of 1. (Note that $\pi(g)$ is automatically diagonalizable, thanks to the hypothesis on k.) The second part follows since the eigenvalues of $\pi(g^{-1})$ are the inverses of those of $\pi(g)$, each such inverse being equal to its conjugate. The third part follows since the trace of a block diagonal matrix is the sum of the traces of the blocks. Finally, the last part follows from the second one, since any matrix has the same trace as its transpose. □

We now study the ij entries $\pi(g)_{ij}$ of the matrices in a representation π in more detail. Let V, W be inequivalent irreducible G-modules, $\pi : G \to GL(V), \mu : G \to GL(W)$ the corresponding representations. Fix bases $v_1, \ldots, v_r$ and $w_1, \ldots, w_s$ of V, W, respectively. For indices i, j with $1 \leq i \leq s, 1 \leq j \leq r$, let T_{ij} be the linear transformation from V to W whose matrix with respect to the given bases of V and W is the matrix unit e_{ij} having 1 in the ijth entry and all other entries 0. Also for $1 \leq i, j \leq r$ let S_{ij} be the linear transformation from V to itself whose matrix relative to the basis of v_i is e_{ij}.

Lemma

We have $\frac{1}{n}\sum_{g\in G}\pi(g)_{ii}\mu(g^{-1})_{jj} = 0$ for all indices i, j, while $\frac{1}{n}\sum_{g\in G}\pi(g)_{ii}\pi(g^{-1})_{jj} = \frac{\delta_{ij}}{r}$, the Kronecker delta divided by r, where n is the order of G.

Proof.

Averaging T_{ij} over G as in the proof of Maschke's Theorem, we get a G-module map from V to W, which must be the 0 map; taking ij-entries, we get the first sum. Similarly, averaging S_{ij} over G, we get a scalar matrix by Schur's Lemma; taking traces, we see that this matrix must be the $1/r$ times the identity if $i = j$ and 0 otherwise. Taking ij-entries, we get the second sum. □

Note that the matrix entries here depend on the choice of bases; but the lemma holds for any such choice. Adding up diagonal entries to compute the trace, we get

Theorem 15: Schur orthogonality I, p. 872

Let χ, χ' be the characters of irreducible representations π, π' of G. Then $\frac{1}{n} \sum_{g \in G} \chi(g)\overline{\chi'(g)} = 1$ or 0, according as π and π' are equivalent or not.

Thus the irreducible characters form an orthonormal basis of the space of (complex-valued) class functions on G (constant on conjugacy classes) with respect to the positive definite Hermitian inner product $(\cdot, \cdot)$ defined by $(c, d) = \frac{1}{n} \sum_{g \in G} c(g)\overline{d(g)}$. For this reason any class function c on G is called a virtual character, being a unique linear combination $\sum_\chi n_\chi \chi$ of irreducible characters χ; we recover the coefficient n_χ of χ in c as the inner product (c, χ). In particular, if c is the character of a representation V, then V is the direct sum of (c, χ) copies of the representation χ corresponding to χ, for all irreducible characters χ.

As a corollary, and as promised above, the character of a representation determines the representation up to equivalence (p. 869). Also a representation V is irreducible if and only if its character χ has square length 1, so that $(\chi, \chi) = 1$. This follows since if V is the direct sum of irreducible subrepresentations $V_1, \ldots, V_m$, then its character is the sum of the characters of the V_i, each having square length one, whence V has square length m.

It is natural to gather the values of the irreducible characters of a group G into a table. In constructing such a table, it is helpful to recall that characters are class functions, so one need only record their values on one element of every conjugacy class. Now let G have m conjugacy classes $C_1, \ldots, C_m$ and let c_i be the size of the ith conjugacy class C_i. Form an $m \times m$ matrix M with rows indexed by irreducible characters χ_i and columns by conjugacy classes C_j, such that the ij entry is the common value of χ on all elements of C_j. Multiplying this entry by $\frac{\sqrt{c_i}}{\sqrt{n}}$ to make a new matrix U, we see that U becomes a matrix with orthonormal rows relative to the standard Hermitian inner product $(\cdot, \cdot)$ on $\mathbb{C}^m$ (such that $(v, w) = \sum_i v_i \overline{w_i}$ if $v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_m)$).

But a matrix U with orthonormal rows is one such that $\overline{U}^t = U^{-1}$, whence any such matrix also has orthonormal columns. Thus we get a second orthogonality relation:

Theorem: 16 Schur orthogonality II, p. 872

For any $g, h \in G$ the sum $\sum_\chi \chi(g)\overline{\chi}(h)$ equals $\frac{n}{c_i}$ or 0, according as g, h lie in the same conjugacy class C_i of G or not; here the sum takes place over the irreducible characters χ of G.

The *original* matrix M, often supplemented by labels of the rows and columns, is called the character table of G (p. 880).

As an example, I compute the character table of the symmetric group $G = S_3$ (see p. 881). I have already observed that G has three irreducible representations, of degrees 1,1, and 2. The first of these is the trivial representation mentioned before, mapping all elements of G to the scalar 1; the second is the sign representation, for which even permutations act by 1 and odd ones by -1. The last representation is often called the reflection representation, acting on $\mathbb{C}^2$ vis the symmetries of an equilateral triangle. Denote the corresponding characters by χ_1, χ_s, and χ_r, respectively. Likewise, G has three conjugacy classes, represented by the identity 1, the transposition (12) flipping the indices 1 and 2, and the 3-cycle (123). The sizes of the classes are 1,3, and 2, respectively. In view of the orthogonality relations, one is rapidly led to the following table.

	1	(12)	(123)
χ_1	1	1	1
χ_s	1	-1	1
χ_r	2	0	-1

As another example with a different flavor, take $G = A_4$, the alternating group on four letters (see p. 883). The elements of order 2 in this group, each of them a product of two disjoint transpositions, together with the identity, form a normal subgroup K of order 4 and index 3; the quotient by this subgroup is cyclic of order 3. As a consequence, there are three one-dimensional representations of G, each trivial on K. To label them, fix ω, a primitive cube root of 1 in $\mathbb{C}$. Label the three one-dimensional characters as χ_1, χ_ω, and χ_{ω^2}. Then there is just one remaining character, of degree 3, labelled χ_r. The conjugacy classes are represented by 1. the product (12)(34) of two disjoint transpositions, the 3-cycle (123), and the cycle (132); they have the respective sizes $1, 3, 4, 4$. The character table is then

	1	$(12)(34)$	(123)	(132)
χ_1	1	1	1	1
χ_ω	1	1	ω	ω^2
χ_{ω^2}	1	1	ω^2	ω
χ_r	3	-1	0	0