Lecture 11-27: Hook formula and symmetric functions

November 27, 2024

The final week of the course will be devoted entirely to review; in this last lecture on new material, I will say a few words about the relationship between Schur polynomials and the characters of Specht modules constructed in Chapter 7.

I begin by observing that the Schur polynomial $s_{\lambda.m}$ indeed depends on the number m of variables as well as the partition λ. For example, taking $\lambda = (2, 1)$, one has $s_{\lambda,1} = 0$, since there are no semistandard tableaux on the set [1] of shape (2, 1); but $s_{(2,1),2} \neq 0$. Fortunately, there is a very simple relationship between $s_{\lambda,m)}$ and $s_{\lambda.n}$ for $n > m$: one has $s_{\lambda,m}(x_1, \ldots, x_m) = s_{\lambda,n}(x_1, \ldots, x_m, 0, \ldots, 0)$. Thus there is what is called a symmetric function s_λ attached to λ alone; this is the family of functions $s_{\lambda,1}, s_{\lambda,2}, \ldots$). In general a symmetric function f is a family of polynomials $(f_1, f_2, \ldots)$, where each f_i is symmetric in $x_1, \ldots, x_i$ and $f_j(x_1, \ldots, x_i, 0, \ldots, 0) = f_i(x_1, \ldots, x_i)$ (p. 77). If in addition all the f_i are homogeneous of the same degree n, then one says that f is homogeneous of degree n as well.

Since the product of two homogeneous symmetric functions of degrees n and m is easily seen to be homogeneous symmetric of degree $n+m$, one has a grading $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$ of the ring Λ of all symmetric functions (say with integer coefficients, taking $\Lambda_0 = \mathbb{Z}$).

The motivation for introducing the ring Λ is that it turns out to be isomorphic to another ring that ties together the representations of all symmetric groups S_n in a beautiful way. To see how this works, let R_n be the free $\mathbb{Z}$-module spanned by the irreducible characters of S_n. Make the direct sum $R = \bigoplus_{n=0}^{\infty} R_n$ into a graded ring by decreeing that the product $\chi \circ \mu$ of the irreducible characters χ, μ of S_m, S_n, respectively, be the induced character $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi \times \mu$ of S_{m+n}, where $\chi \times \mu$ is regarded as a character of the direct product $S_m \times S_n$ via $\chi \times \mu(g, h) = \chi(g)\mu(h)$ and $S_m \times S_n$ is embedded in S_{m+n} in the obvious way. A straightforward argument using the transitivity of induction (Proposition 14 on p. 898 of Dummit and Foote) shows that multiplication in R is associative and commutative.

Then one has

Theorem, p. 91

The graded rings R and Λ are isomorphic by the map sending the character χ_λ of the Specht module S^λ (lying in R_n) to $s_\lambda \in \Lambda_n$.

This result is particularly remarkable since characters of Specht modules (class functions on symmetric groups) look nothing like symmetric functions or polynomials. Also semistandard tableaux with repeated entries contribute terms to Schur functions, but not to characters of symmetric groups.

One can then deduce the following restriction formula for representations of S_n.

Theorem, p. 93

Given a partition λ of n, the restriction of the Specht module S^λ to S_{n-1} as the sum $\sum S^{\lambda'}$ as λ' runs through the partitions whose diagrams are obtained from that of λ by deleting one box. In particular, this restriction is irreducible if and only if the diagram of λ is a rectangle (all rows have the same length).

If one restricts S^λ to S_{n-1}, then to S_{n-2}, and so on, all the way to S_1, one obtains the sum of $f^\lambda = \dim S^\lambda$ copies of the trivial representation. So f^λ equals the number of ways to reduce the diagram D_λ of λ to a single box by removing one box at a time, making sure that resulting shape is always a diagram. Now the boxes that can be removed from D_λ at the first step are exactly those which can be filled by the largest number n in a standard tableau of shape λ, and similarly for the boxes that can be removed at the subsequent steps.

In this way we recover the earlier result that dim S^λ equals the number of standard tableaux of shape λ. I conclude with a beautiful formula due to Frame, Robinson, and Thrall, for the number of standard tableaux of shape λ, where λ is a partition of n. To state it, I need some terminology.

Given a Young diagram, each box in it corresponds to a hook, consisting of itself and all boxes either directly below it or directly to its right; its length is the number of boxes in it. For example, starting with the partition $\lambda = (3, 2, 1)$, I have labelled each box in the corresponding Young diagram by the length of its hook below.

$$\begin{matrix} 5 & 3 & 1 \\ 3 & 1 & \\ 1 & & \end{matrix}$$

Then we have

Hook length formula

If $n = |\lambda|$, then the number of standard tableaux of shape λ equals $n!$ divided by the product of the hook lengths in the Young diagram of λ.

Thus for example we have $\dim S^{(3,2,1)} = \frac{6!}{5\cdot 3\cdot 1\cdot 3\cdot 1} = 16$.

Many proofs of this result have been given since the original one in 1954, which used the representation theory of S_n in prime characteristic. A straightforward proof by induction is given in Chapter 4 of Fulton's book, together with a heuristic argument which can be converted to a rigorous probabilistic proof. More recently Pak and others have given a geometric proof, by computing the n-dimensional volume of a certain polytope.