# Lecture 11-25: Representations of $GL_n(\mathbb{C})$

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I now specialize down to the case where the ring $R$ of last time is the complex numbers and the module $M$ is $\mathbb{C}^m$, so that the group $G = GL_m(\mathbb{C})$.

Taking $R = \mathbb{C}, M = \mathbb{C}^m$, I begin by observing that the representations $M^\lambda$ from last time are polynomial in the sense that the formulas for the entries of $\pi(x)$ in the corresponding homomorphisms $\pi$ from $G$ to the appropriate $GL_n(\mathbb{C})$ are given by polynomial functions in the entries of $x \in G$. Indeed, this is clear for $M^1 = M$ itself; it follows for $M^\lambda$ in general from the explicit construction of this module.

To get a picture of what a general $M^\lambda$ looks like, let $d \in G$ be a diagonal matrix, say with diagonal entries $d_1, \ldots, d_m$. Then $de_i = d_i e_i$ for the $i$th unit coordinate vector $e_i$; since $d$ acts on a tableau $T \in V = M^\lambda$ by simultaneously acting on all of its entries, one sees that if the entries of $T$ are numbers (corresponding to unit coordinate vectors), then $dT = \prod_{i=1}^m d_i^{n_i} T$, where $n_i$ is the number of times the integer $i$ occurs as an entry in $T$. In particular, $d$ acts diagonally on $V$ with eigenvalues $\prod_{i=1,T}^m d_i^{n_{i,T}}$ as $T$ runs through all semistandard tableaux of shape $\lambda$ with entries in $[m]$ (call these tableaux on $[m]$ for short) and $n_{i,T}$ is the number of times $i$ occurs in $T$. Each eigenvalue $d_1^{a_1} \ldots d_m^{a_m}$ occurs as many times as there are semistandard tableaux of shape $\lambda$ with exactly $a_i$ is for all $i$.

One therefore defines the Schur polynomial $s_{\lambda,m}$ in the variables $x_1, \ldots, x_m$ by the formula $\sum_T x^T$, where $T$ runs over semistandard tableaux of shape $\lambda$ on $[m]$. Here $x^T$ denotes $x_1^{a_1} \ldots x_m^{a_m}$, where $T$ has $a_i$ $i$s for $1 \leq i \leq m$ (p. 3). You can think of $M^\lambda$ as a kind of "country" with "cities" corresponding to tuples $(a_1, \ldots, a_m)$; the "population" of the city corresponding to this tuple is the coefficient of $x_1^{a_1} \ldots x_m^{a_m}$ in $s_{\lambda,m}$.

When semistandard tableaux $T$ of shape $\lambda$ on $[m]$ are regarded as vectors in $M^\lambda$ they are called weight vectors, by virtue of being eigenvectors for all diagonal matrices in $G$ simultaneously; their weights, regarded as functions on the subgroup of diagonal matrices in $G$, are monomials in the diagonal entries $x_1, \ldots, x_m$ of these matrices. One of these weight vectors $v_H$, corresponding to the tableau $H$ with all 1s in the first row, all 2s in the second, and so on, is especially important: if $u \in G$ is upper triangular, then repeated use of multilinearity and the alternating relations shows that $uv$ is scalar multiple of $v$. This vector $v$ is called a highest weight vector, of highest weight $w = x_1^{\lambda_1} \ldots x_m^{\lambda_m}$, where $\lambda_1, \ldots, \lambda_m$ are the parts of $\lambda$; here we allow some of these parts to be 0 (but recall that the representation $M^\lambda = 0$ if $\lambda$ has more than $m$ parts).

The reason that the weight $x_1^{\lambda_1} \dots x_m^{\lambda_m}$ is called "highest" is that every other weight $x_1^{\mu_1} \dots x_m^{\mu_m}$ occurring in $M^\lambda$ is easily seen to satisfy $\mu_1 \leq \lambda_1, \mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$, and so on, so that there is a partial order on the weights occurring in $M^\lambda$ such that $w$ is the unique maximal weight in this order. Moreover, the only semistandard tableau of shape $\lambda$ contributing the monomial $w$ to the Schur polynomial $s_{\lambda,m}$ is $H$, so that this monomial occurs with coefficient 1 in $s_\lambda$. Furthermore, one can check that the only tableau $T$ of shape $\lambda$ with $uT$ a scalar multiple of $T$ for all upper triangular matrices $u$ is $H$ itself, so that up to scalar multiples $v_H$ is the only highest weight vector in $M^\lambda$.

Now it turns out that by the general theory of finite-dimensional modules over the (complex semisimple) Lie (or linear algebraic) group $G$, any such module is a direct sum of irreducible modules, and every irreducible module has a unique highest weight vector up to scalar multiple. Thus the modules $M^\lambda$ are all irreducible, and in fact up to isomorphism they are the only finite-dimensional irreducible polynomial representations of $G$. Using language that I introduced in the lecture on November 1, $G$ has tame representation type: it has infinitely many inequivalent irreducible(=indecomposable) modules, but these can be parametrized in a nice way (by partitions with at most $m$ parts). In particular, there are only countably many of them.

Last time I observed that there is an injective map from $M^\lambda$ to a certain polynomial ring $R[X]$; since this ring is independent of $\lambda$, one can exploit the multiplication structure on it to put a multiplication structure on the direct sum of all $M^\lambda$. This sum turns out to be closely connected to the geometry of a certain quotient of $G$. I will return to Fulton's book to say more about this connection in the spring, when I treat commutative algebra. I will also say more about multiplication of Schur functions below.

The definition of the Schur polynomials $s_{\lambda,m}$ given above makes no direct reference to representation theory, requiring as it does only the definition of semistandard tableau on $[m]$. It turns out that these polynomials are of considerable interest even outside representation theory. This is primarily because they turn out to be symmetric polynomials, remaining unchanged if the variables $x_1, \ldots, x_m$ are permuted.

For example, if $\lambda$ has a single part $r$, then the Schur polynomial $s_{\lambda,m}$ is the sum of all products of $r$ not necessarily distinct variables among $x_1, \ldots, x_m$, a famous symmetric polynomial called the $r$th complete symmetric polynomial in $m$ variables and denoted $h_r(x_1, \ldots, x_m)$ (p. 72). If instead $\lambda = (1, \ldots, 1)$ has $r$ parts, all equal to 1, then $s_{\lambda,m}$ is the sum of all products of $r$ *distinct* variables $x_1, \ldots, x_m$; this polynomial is called the $r$th elementary symmetric polynomial in $m$ variables and denoted $e_r(x_1, \ldots, x_m)$ (p. 72). Note that this last polynomial is 0 whenever $r > m$.

In general there is a beautiful formula for $s_{\lambda,m}$ as the determinant of a matrix whose entries are complete (or alternatively elementary) symmetric polynomials (see p. 75), so that $s_{\lambda,m}$ is indeed symmetric; it is also homogeneous of degree equal to the sum $n$ of the parts of $\lambda$. I will give an alternative proof of the symmetry of these polynomials below.

First recall that if $V, W$ are modules for a group $G$ over the same field $K$, then the tensor product $V \otimes_K W$ as a module for $G$ via the recipe $g(v \otimes w) = gv \otimes gw$. More generally, if $V, W$ are modules for the respective groups $G, H$ over the same field $K$, then $V \otimes_K W$ becomes a module over the direct product $G \times H$ via the recipe $(g, h)(v \otimes w) = gv \otimes hw$. (The previous recipe is just a restriction of this one to the diagonal copy of $G$ in $G \times G$, consisting of all order pairs $(g, g)$ with $g \in G$). It is not difficult to check that if $\chi_V, \chi_W$ are the respective characters of $V, W$ on $G, H$, then the character of $V \otimes_K W$ is the product $\chi_V \chi_W$, whose value at $(g, h) \in G \times H$ is just $\chi_V(g)\chi_W(h)$.

The notion of character extends from finite groups to $G = GL_m(\mathbb{C})$, as follows. I take the character of a representation $\pi : G \to GL_n(\mathbb{C})$ to be the function sending $g \in G$ to the trace of $\pi(g)$, as before; but it is convenient to consider only diagonal matrices $g$. If the diagonal entries of $g$ are $x_1, \ldots, x_m$, then the trace of $g$ on $M^\lambda$ is easily seen by the above remarks to be the Schur polynomial $s_{\lambda,m}$ (the sum of its eigenvalues). In particular, since permuting the $x_i$ gives another diagonal matrix similar to the first, it must have the same trace, so that $s_{\lambda,m}$ is indeed symmetric.

The subring of polynomial functions in $x_1, \ldots, x_m$ generated by the characters $s_{\lambda.m}$ then turns out to consist precisely of the symmetric polynomials in these variables (p. 123); it is equivalent to say that the polynomials $s_{\lambda,m}$ provide a basis for the symmetric polynomials in the $x_i$. In particular, thanks to the above remarks about tensor products, the product $s_{\lambda,m}s_{\lambda',m}$ is a nonnegative integral combination of $s_{\mu,m}$ for any partitions $\lambda, \lambda'$ with at most $m$ parts. There is a purely combinatorial recipe for computing the coefficients in this combination called the Littlewood-Richardson Rule, which I hope some of you will have occasion to learn in the future. There is an account of this rule in Chapter 5 of Fulton.