Lecture 11-22: Representations of the general linear group

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I round out the course with representations of the (in general infinite) general linear group $G = GL_m(R)$ of all invertible $m \times m$ matrices, over a commutative ring R, giving a construction of these very similar to the one I gave of representations of symmetric groups. I will follow Chapter 8 of Fulton.

I begin with a very general construction, which given an R-module M and a partition λ of n produces a new R-module denoted $M^\lambda$and called the Schur module of M and λ (see p. 106). Start with the n-fold tensor power $\otimes^n M$, writing its decomposable elements as Young diagrams T of shape λ whose boxes are filled with elements of M rather than integers as before. Generalizing earlier terminology, call these tableaux.

Impose the alternating relations $T = -T'$ whenever T' is obtained from T by interchanging two elements in the same column, while $T = 0$ if two elements in the same column of T coincide. Next impose the exchange relations $T = \sum_{i=1}^{r} T_i$, where c_1, c_2 are two columns of Y with c_2 to the right of c_1, k is an integer at most equal to the length of c_2, and $T_1, \ldots T_r$ are the tableaux obtained from T by interchanging the top k entries of c_2 with all possible sets of k entries of c_1, preserving the vertical order of entries throughout.

Now specialize down to the case where $M = R^m$ is free over R. Observe that the defining relations of M^λ in this case are very similar to those of the Specht module S^λ derived last time. To strengthen this analogy, note that $G = GL_m(R)$ acts in a natural way on any tableau in M^λ, by acting on each entry simultaneously, and that this action preserves the defining relations of both the tensor power and M^λ. Thus M^λ is a representation of G for every partition λ, which by multilinearity is spanned by tableaux T of shape λ in which the entries are standard basis vectors e_i of R^m for $1 \leq i \leq m$. Rewriting T by replacing e_i by i throughout, one finds that the T becomes a Young tableau in the earlier sense.

Unlike S^λ, however, M^λ is not spanned by standard tableaux of shape λ, but rather by semistandard tableaux, whose entries once again increase strictly down columns, but this time increase only weakly across rows. In particular $M^\lambda = 0$ if λ has more than m parts, since then there are no semistandard tableaux of shape λ with entries in $[m] = \{1, \ldots, m\}$, and indeed the alternating relations and multilinearity show that every tableau of shape λ in this case equals 0 in M^λ.

In the special case where $\lambda = r$ has just a single part, the exchange relations show that M^λ identifies with the rth symmetric power $S^r R^m$, which I showed in the lecture on October 11 to have a basis over R corresponding to the semistandard tableaux of this shape (equivalent to monomials in a set of m variables). Similarly, if $\lambda = (1, \ldots, 1)$ is such that its diagram has a single column, say of length r, then M^λ identifies with the exterior power $\bigwedge^r \mathbb{R}^m$ by the alternating relations, and again you have seen that M^λ is spanned by semistandard tableaux in this case. To see what is going on in general, I digress to prove a key result in linear algebra.

Sylvester's Lemma, p. 108

For any $p \times p$ matrices M, N over R and an index k between 1 and p, one has

$$\det(M)\det(N) = \sum \det(M')\det(N')$$

where the sum takes place over all pairs (M', N') of matrices obtained from (M, N) by interchanging a fixed set of k columns in N with any k columns of M, preserving the positions of the columns.

Proof.

By the alternating property of determinants I may assume that the fixed set of columns of N consists of its leftmost k columns. For vectors $v_1, \ldots, v_p \in R^p$, denote by $[v_1 \ldots, v_p]$ the determinant of the matrix having the v_i as its columns. I must show that

$$[v_1 \ldots, v_p][w_1 \ldots w_p] = \sum_{i_1 < \cdots < i_k} [v_1 \ldots w_1 \ldots w_k \ldots v_p][v_{i_1}, \ldots v_{i_k}, w_{k+1} ..]$$

where $w_1, \ldots, w_k$ replace $v_{i_1}, \ldots, v_{i_k}$, in that order, in the leftmost factor. It suffices to show that the difference of the two sides is an alternating function of the $p+1$ vectors $v_1, \ldots, v_p, w_1$, regarding the other vectors as fixed, since any such function must be identically 0. □

Proof.

For this it suffices to show that the two sides are equal when two consecutive vectors $v_i, .v_{i+1}$ are equal, which is immediate, and when $v_p = w_1$. In the latter case, fixing $v_p = w_1$, it suffices to show that the difference of the two sides is an alternating function of $v_1, \ldots, v_p, w_2$. Now the cases $v_i = v_{i+1}$ and $v_p = w_2$ are both immediate. □

Given positive integers m, n let x_{ij} for $1 \leq i \leq m, 1 \leq j \leq n$ be independent variables over R and let $\mathbb{R}[X]$ denote the polynomial ring over R in these variables. For each p-tuple $(i_1, \ldots, i_p)$ of integers from $\{1, \ldots, m\}$ set

$$M_{i_1 \ldots i_p} = \det \begin{bmatrix} x_{1i_1} & \ldots & x_{1i_p} \\ \vdots & & \vdots \\ x_{pi_1} & \ldots & x_{pi_p} \end{bmatrix}$$

and let $D_{i_1 \ldots i_p} \in R[X]$ be its determinant, an alternating function of the subscripts. For a tableau T with at most m rows filled with entries in $\{1 \ldots, m\}$, let D_T be the product of the determinants corresponding to the columns of T, so that

$$D_T = \prod_{j=1}^{\ell} D_{T(1,j),T(2,j)\ldots,T(\mu_j,j)},$$

where μ_j is the length of the jth column of T, ℓ is the number of columns of T, and $T(i,j)$ is the entry of T in its ith row and jth column.

Lemma 3, p. 109

If $M = R^m$ is free over R then there is a canonical homomorphism from M^λ to $\mathbb{R}[X]$ sending T to D_T for all tableaux T with entries in $[m]$.

It suffices to show that the D_T satisfy the alternating and exchange relations. The alternating relations follow from the alternating property of determinants. Given two columns of T between which an exchange takes place, with entries $i_1, \ldots, i_p$ in the leftmost column and $j_1, \ldots, j_q$ in the rightmost column, an application of Sylvester's Lemma to the matrices $M = M_{i_1,\ldots,i_p}$ and N, obtained from $M_{j_1,\ldots,j_q}$ by adding a $q \times (p-q)$ block of zeroes to its right on top of a copy of the identity matrix of size $(p-q) \times (p-q)$ as its new lower right corner, satisfies the corresponding exchange relation.

Theorem 1, p. 110

With $M = R^m$ as above the module M^λ is free with basis consisting of all (classes of) semistandard tableaux T with entries in $[m]$.

To prove this I argue as in the proof of Lemma 6 on p. 98, given last time. Given two tableaux T, T', I once again decree that $T \succ T'$ if in the rightmost column where they are different, the lowest box where they differ has a larger entry in T'. Given any $T \in M^\lambda$ that is not semistandard, I again invoke the alternating and exchange relations to write it as a linear combination of tableaux T' that are higher in this order. This shows that the classes of standard T generate M^λ.

To show that these classes are linearly independent, I invoke Lemma 3 above, which implies that it is enough to show that the determinants D_T are independent in $R[X]$ as T runs over semistandard tableaux. Order the variables x_{ij} lexicographically, so that $x_{ij} < x_{i'j'}$ if $i < i'$ or $i = i'$ and $j < j'$. Extend this lexicographic order to monomials M_i in the x_{ij}, so that $M_1 < M_2$ if the smallest x_{ij} that occurs to a different power in the M_i occurs to a larger power in M_1. Note that if $M_1 < M_2$ and $N_1 \leq N_2$ then $M_1 N_1 < M_2 N_2$. It follows at once that the smallest monomial appearing in a determinant $D_{i_1 \ldots i_p}$ is the diagonal term $x_{1,i_1} x_{2,i_2} \ldots x_{p,i_p}$, if $i_1 < \ldots < i_p$.

Then the smallest monomial occurring in D_T, if T has increasing columns, is $\prod (x_{ij})^{m_T(i,j)}$, where $m_T(i,j)$ is the number of times j occurs in the ith row of T. This monomial occurs with coefficient 1. Now reorder the tableaux, decreeing that $T < T'$ if the first row where they differ, and the first entry where they differ in that row, is smaller in T than in T'. Equivalently, the smallest i for which there is a j with $m_T(i,j) \neq m_{T'}(i,j)$, and the smallest such j, has $m_T(i,j) > m_{T'}(i,j)$. It follows that if $T < T'$ then the smallest monomial occurring in D_T is smaller than any monomial occurring in $D_{T'}$. Now the independence follows: if $\sum r_T D_T = 0$, take T minimal with $r_T \neq 0$; then the coefficient of $\prod (x_{ij})^{m_T(i,j)}$ in $\sum r_T D_T$ is r_T. □

As a corollary of the proof, I note that the map from M^λ to $R[X]$ is injective and its image is free as an R-module on the set of polynomials D_T as T runs over the semistandard tableaux of shape λ. Next time I will specialize to the case $R = \mathbb{C}, M = \mathbb{C}^m$; in this case the representations M^λ of $GL_m(\mathbb{C})$ turn out to be irreducible and to exhaust the set of so-called polynomial representations of that group.