# Lecture 11-15: Induced characters and representations

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Given a representation $V$ of a group $G$, one can clearly restrict it to a subgroup $H$. Its character is then just the restriction to $H$ of the character $\chi$ of $V$. It turns out that there is an operation called *induction* going the other way, from representations or characters of $H$ to representations or characters of $G$. This turns out to be a very useful way of obtaining representations of larger groups from those of smaller ones.

## Definition, p. 893

Given an $H$-module $V$, the *induced representation* (or module) $W = \mathrm{Ind}_H^G V$ is $\mathbb{C}G \otimes_{\mathbb{C}H} V$, regarded as a $\mathbb{C}G$-module by virtue of the $(\mathbb{C}G, \mathbb{C}H)$-bimodule structure of $\mathbb{C}G$.

Since $\mathbb{C}G$ is free over $\mathbb{C}H$, a basis for $W$ is given by the tensors $g_i \otimes v_j$ as the $v_j$ run over a basis of $V$ and $g_i$ run over a set of representatives for the left cosets of $H$ in $G$, In particular, $\dim W = [G : H] \dim V$.

Multiplication by $g \in G$ on the subspace $x \otimes V$ of $W$ sends it to the subspace $gx \otimes V$, which coincides with $x \otimes V$ if and only if $x^{-1}gx \in H$. It follows that the character $\chi_G$ of $\mathrm{Ind}_H^G V$ is given by the formula $\chi_G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_V(x^{-1}gx)$, where $\chi_V$ is the character of $V$ (seeTheorem 11, p. 893). We say that $\chi_G$ is induced from the character $\chi_V$. More generally

## Definition: Corollary 12, p. 894

Given a class function $c_H$ on $H$, define $c_G = \mathrm{Ind}_H^G c_H$, the function induced from $c_H$, via $c_G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} c_H(x^{-1}gx)$, where $|H|$ denotes the order of $H$. Equivalently, letting $g_1, \ldots, g_m$ to be a set of representatives of the left cosets of $H$ in $G$, take $c_G(g) = \sum_{g_i^{-1}xg_i \in H} c_H(g_i^{-1}xg_i)$.

A representation induced from an irreducible one of a smaller group need not be irreducible, but there is a simple formula for coefficient of any character in an induced character. This is

**Theorem: Frobenius reciprocity; see Exercise 14, p. 904**

With notation as above, one has $(c_G, \rho) = (c_H, \rho_H)$ for all characters $\rho$ of $G$, where $\rho_H$ denotes the restriction of $\rho$ to $H$.

**Proof.**

By definition

$$(c_G, \rho) = \frac{1}{|G|} \sum_{g \in G} c_G(g)\overline{\rho(g)} = \frac{1}{|G|}\frac{1}{|H|} \sum_{g \in G} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} c_H(x^{-1}gx)\overline{\rho(x^{-1}gx)}.$$

In this last sum every term $c_H(h)\overline{\rho(h)}$ appears exactly $|G|$ times for each fixed $h \in H$, whence the sum equals $(c_H, \rho_H)$, as claimed. □

This last result is usually expressed by saying that induction from $H$ to $G$ is the left adjoint of restriction from $G$ to $H$: one moves from the left to the right side of the inner product by replacing an induced representation by a restricted one. This kind of adjointness appeared previously in conjunction with homomorphisms and the tensor product; see Theorem 43 on p. 401.

As an example, take $G = S_3$ and let $H$ be the cyclic subgroup generated by a transposition. Inducing the trivial character from $H$ to $G$, I get a character taking the value 3 at the identity $e$, 1 on a transposition, and 0 on a 3-cycle; taking the square length of this character, I find that it is the sum of two irreducible characters. Subtracting off the trivial character (which occurs in it by Frobenius reciprocity) I get the character denoted by $\chi_r$ in a previous lecture, taking the value 2 at $e$, 0 on a transposition, and $-1$ on a 3-cycle.

Thus if I had never heard of the representation with this character, I could reconstruct it by induction. A similar calculation shows that inducing the nontrivial character from $H$ to $G$ gives the character with value 3 at $e$, $-1$ on a transposition, and 0 on a 3-cycle; subtracting off the character $\chi_r$ just constructed, i recover the character of the sign representation, which is 1 on $e$ and a 3-cycle and $-1$ on a transposition.

One can also get interesting information by inducing class functions that are not characters. As an example, start with the cyclic subgroup $T$ generated by a 3-cycle in the alternating group $A_4$ on four letters. Take each of the one-dimensional characters of $T$, subtract off the trivial character, and induce the resulting class function to $A_4$. You get $\chi_1 - \chi_1, \chi_2 - \chi_1, \chi_3 - \chi_1$, where the $\chi_i$ range over the three one-dimensional characters of $A_4$ (constructed previously; here $\chi_1$ is the trivial character). What is going on in both of these examples is that there is a subgroup $C$ of $G$, equal to $H$ in the first example and $T$ in the second, which is such that $g^{-1}Cg \cap C = 1$ for any $g \in G$ with $g \notin C$.

More generally, let $G$ be a transitive subgroup of the symmetric group $S_n$ (acting on the index set $\{1, \ldots, n\}$ with a single orbit) such that only the identity in $G$ fixes as many as two indices among $1, \ldots, n$. Let $H$ be the stabilizer in $G$ of any index, say 1. The hypothesis on $G$ implies that $H \cap g^{-1}Hg = 1$ whenever $g \notin H$. A famous result of Frobenius then asserts that the set of permutations in $G$ fixing no index, together with the identity, is a normal subgroup $N$ of $G$ and $G = HN$; this is called the *Frobenius kernel* while the subgroup $H$ is called the *Frobenius complement*. You will prove this in homework next week, using the theory of induced characters.

As a final example, the dihedral group $D_n$ of an $n$-gon with $n$ odd also satisfies the above hypothesis, with $H$ the subgroup generated by a single reflection and $N$ the subgroup of rotations.