Lecture 11-1: Irreducible and indecomposable representations

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Last time I showed that there are exactly p equivalence classes of indecomposable representations of the cyclic group C_p of prime order p over any field k of characteristic p; these correspond to the Jordan blocks of size at most $p \times p$ and eigenvalue 1. One describes this situation by saying that the cyclic group C_p has finite representation type over any field of characteristic p. Indecomposable representations of the *infinite* cyclic group $\mathbb{Z}$ over any field k correspond up to equivalence to companion matrices of powers of irreducible polynomials other than x over k; thus there are infinitely many of them, but they can be parametrized in a nice way. One says that $\mathbb{Z}$ has tame representation type.

In stark contrast, indecomposable representations of the Klein four-group $C_2 \times C_2$ in characteristic 2 do not admit any reasonable parametrization; entire Ph.D. theses and papers are written on such representations. We say that $C_2 \times C_2$ has wild representation type (in characteristic 2). We can get a better handle on representations if we replace indecomposability by a stronger hypothesis.

Definition, p. 847

The representation V of G is called *irreducible* or *simple* if it does not admit any proper subrepresentation.

For example, any one-dimensional representation is trivially irreducible. For any irreducible polynomial p over a field k, we have already observed that the quotient $R = k[x]/(p)$ does not admit any proper subspace stable under multiplication by x, since any such subspace would correspond to a proper ideal of R as a ring. Thus in particular the representation $\mathbb{Q}[x]/(\Phi_n)$ of the cyclic group C_n over $\mathbb{Q}$ is irreducible, where Φ_n is the nth cyclotomic polynomial, since I will define this polynomial next term and show that it is irreducible. On the other hand, any representation of C_n over a field k for which the polynomial $x^n - 1$ is a product of distinct linear factors is a direct sum of one-dimensional subrepresentations, so that the only irreducible representations of C_n over any such k are one-dimensional, with the generator g of C_n acting by an nth root of 1 in k.

More generally, one has

Theorem; see Exercise 19, p. 854

Let A be a finite abelian group of order n and k a field with n distinct nth roots of 1 (so that in particular the characteristic of k does not divide n). Then any representation V of A is a direct sum of one-dimensional representations, so that every irreducible representation is one-dimensional. There are n inequivalent irreducible representations of A.

Proof.

We know that A is isomorphic to a direct sum $\bigoplus_{i=1}^{m} C_{n_i}$ of cyclic groups with orders n_i dividing n. Letting g_i be a generator of C_{n_i}, we have seen that V is the direct sum of eigenspaces E of g_i, each with eigenvalue e_i, an n_ith root of 1 in k. Since A is abelian, every eigenspace E is stable under the action of the generators g_j with $j \neq i$ of the other cyclic factors C_{n_j} of A. By induction on the number of cyclic factors, we deduce that V is the direct sum of one-dimensional simultaneous eigenspaces of all generators g_i. In particular all irreducible representations are one-dimensional. Conversely, given a one-dimensional space V over k, we make it into a representation by decreeing that each generator g_i act by a scalar equal to a suitable e_i; as there n_i choices for each e_i and the product of the n_i is n, the result follows. □

In fact the set $\widehat{A}$ of equivalence classes of irreducible representations of A over k has a group structure and $\widehat{A} \cong A$ (cf. Exercise 16, p. 853). This holds because such classes correspond to homomorphisms from A to $GL_1(k) = k^*$ and the product $\pi_1\pi_2$ of two such homomorphisms π_1, π_2 (sending $a \in A$ to $\pi(1)\pi_2(a)$) is another homomorphism. Since (as noted above) any π sends a generator g_i of a cyclic factor of A to an n_ith root of 1 in k, the isomorphism follows. Note however that there is no *canonical* homomorphism from $\hat{A}$ to A, since the isomorphism between these groups depends on the choices of a particular roots of 1 in k.

Of course most finite groups G are nonabelian; accordingly most irreducible representations of such groups have degree larger than one. The following key result reduces the study of such representations to the irreducible case, under a mild (and by now familiar) restriction on k.

Maschke's Theorem, p. 849

If the characteristic of k does not divide the order n of G, then any representation V of G over k is semisimple; that is, it is the direct sum of irreducible subrepresentations.

Proof.

There is nothing to prove if V is irreducible, so assume not and let W be a proper subrepresentation. It is enough to show that there is another subrepresentation W' complementary to W in V, so that V is the direct sum of W and W', for then by induction on dimension both W and W' are direct sums of irreducible subrepresentations, whence so is V. To construct W', let f be any linear projection of V onto W (so that f maps V onto W and the restriction of f to W is the identity). Set $\tilde{f}(v) = \frac{1}{n} \sum_{g \in G} g^{-1} f(gv)$.

Then for $h \in G$ we have $h^{-1}\tilde{f}(hv) = \frac{1}{n} \sum_{g \in G} h^{-1} g^{-1} f(ghv) = \tilde{f}(v)$, so that $\tilde{f}$ is a G-module homomorphism from V to W, which is the identity on W, since f is. The kernel of $\tilde{f}$ is then a submodule W' intersecting W trivially; computing its dimension we see that V is the direct sum of W and W', as desired. □

The technique of this proof is called averaging over G and occurs frequently in the study of finite (or more generally compact) groups. We now study irreducible modules in a rather roundabout way, first investigating homomorphisms between them rather than the modules themselves. Note first that the set $\hom_G(M, M')$ of G-module homomorphisms from one module M to another one M' is clearly a vector space over the basefield k.

Theorem: Schur's Lemma; see Exercise 18, p. 853

Assume that k is algebraically closed. Let V, W be irreducible G-modules. If V is not isomorphic to W, then $\hom_G(V, W) = 0$. If V is isomorphic to W, then $\hom_G(V, W) \cong k$.

Proof.

The kernel and image of any G-module homomorphism are both G-submodules, so if $V \not\cong W$, then any module homomorphism from V to W necessarily has kernel V and image 0, by irreducibility. If $V \cong W$ let f be an isomorphism and g a homomorphism between them. The homomorphism $f^{-1}g$ from V to V, as a linear map, must have an eigenvalue λ; but then its λ-eigenspace, being the kernel of $g - \lambda f$, must be a nonzero G-submodule of V and thus all of V. Hence $g = \lambda f$, as claimed. □

This result says in particular that the only linear maps from V to itself commuting with the action of G are the scalars.

We now introduce a ring R such that modules over this ring (for fixed basefield k) are the same things as G-modules.

Definition, p. 840

Given k and G, the *group algebra* kG consists of all formal linear combinations $\sum_{g \in G} k_g g$ of elements of G with coefficients in k. The group elements g are regarded as linearly independent, so that two such combinations agree if and only if their coefficients match up term by term. We add two such combinations and multiply by elements of k in the obvious way. We take the product $\sum_{g \in G} k_g g \sum_{g \in G} \ell_g g$ of two elements of kG to be $\sum_{g,h \in G} k_g \ell_h gh$, collecting coefficients in this last sum to make the group elements appearing in it distinct.

Given any G-module V over k we then make V into a kG-module by decreeing that $(\sum_{g \in G} k_g g)v = \sum_{g \in G} k_g(gv)$.

Equivalently, any homomorphism $\pi : G \to GL(V)$ extends uniquely to a k-algebra homomorphism from kG to the ring $M(V)$ of all linear transformations from V to itself, where V is a finite-dimensional vector space over k. In particular, kG itself, clearly being a kG-module, is also a G-module. We call it the regular representation of G.