Lecture 10-30: Representations of finite groups

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So far I have focussed on similarity classes of matrices. I now in effect pass to similarity classes of certain sets of matrices arising from groups, trying to understand both groups and matrices better by exploiting the connections between them.

Given a group G and a finite-dimensional vector space V over a field k, I define a representation of G to be a homomorphism π from G to the group $GL(V)$ of invertible linear transformations from V to itself (p. 840). The dimension of V is called the degree (or dimension) of π. While this is the official definition of representation, in practice one usually prefers to work with the vector space V rather than the homomorphism π; accordingly it too is often called a representation of G. V can also be called a G-module, since G acts linearly on it: given $g \in G, v \in V$, we define $gv = \pi(g)v$.

For example, the group G of (length and angle-preserving) symmetries of any subset S of $\mathbb{R}^n$ with centroid at the origin turns out to act linearly on $\mathbb{R}^n$, so that this action gives rise to an n-dimensional (real) representation of G.

Two representations $\pi : G \to GL(V), \pi' : G \to GL(W)$ are called equivalent if the G-modules V and W are isomorphic, so that there is a there is an invertible linear map σ mapping V onto W such that $\sigma(\pi(g)v) = \pi'(g)\sigma(v)$ for all $g \in G$ (p. 846). Such a map σ is often called an intertwining operator. If $V = W$, this says exactly that there is an invertible linear map P on V with $\pi'(g) = P\pi(g)P^{-1}$ for all $g \in G$.

If V is a representation of G and W is a subspace of V stable under the action of G, then we call W a subrepresentation of V. If V is the direct sum of two subrepresentations V_1, V_2, then we say that V is decomposable (p. 847). More generally, given two representations V_1, V_2 of G, we make their direct sum $V_1 \oplus V_2$ a representation of G in the obvious way; the degree of $V_1 \oplus V_2$ is the sum of the degrees of V_1 and V_2.

If G is infinite, then it typically has additional structure; at the very least it will almost always be a topological group. In this case we usually insist that a representation π preserve this structure, so that we restrict to continuous π if G is a topological group, to smooth π if G is a Lie group, and to holomorphic π if G is a complex Lie group. In this course we will always assume that G is finite, in which case no additional restrictions are placed on π.

Example

We begin with the simplest possible group G, namely a cyclic group C_n of order n with g as generator. Clearly any representation π of G is determined by the single matrix $\pi(g)$; given the above definition of equivalence, we see that equivalence classes of representations of G of degree m are in natural bijection to similarity classes of $m \times m$ matrices M with $M^n = I$.

Example

The number of such classes depends heavily on how the polynomial $x^n - 1$ factors in $k[x]$. If for example k is the complex field, or more generally any algebraically closed field of characteristic not dividing n, then this polynomial splits over k into distinct linear factors. If instead $k = \mathbb{Q}$, then this polynomial is the product of irreducible cyclotomic polynomials $\Phi_d(x)$ as d runs over the divisors of n, the degree of Φ_d being $\phi(d)$, the Euler ϕ function of d, which counts the number of positive integers less than d and relatively prime to it. See Section 13.6 of the text (pp. 552 and following); I will say a lot more about cyclotomic polynomials next term.

The situation is very different if the characteristic of k divides n, whether or not k is algebraically closed, since then the polynomial $x^n - 1$ fails to have distinct roots. More precisely, if $x^n - 1$ is the product of distinct linear factors in $k[x]$, then all Jordan blocks of $M = \pi(g)$, the image under a representation π of a generator g of C_n, have size 1×1 and eigenvalue an nth root of 1 in k. If on the other hand k has prime characteristic p and $n = p$, then this same image $\pi(g)$ of a generator of C_p has all Jordan blocks of size $\ell \times \ell$ for some $\ell \leq p$ and unique eigenvalue 1. We deduce

Theorem

Over an algebraically closed field k of characteristic not dividing n, any representation V of C_n is the direct sum of one-dimensional representations, on each of which the generator g of C_n acts by an nth root of 1 in k. If $n = p$ is prime and k has characteristic p, then any representation of C_p is a direct sum of representations $k[x]/((x-1)^\ell$ for various $\ell \leq p$, in each of which the matrix $\pi(g)$ representing the generator g is similar to an $\ell \times \ell$ Jordan block with 1s on the diagonal.

See Example 5 on pp. 848-9 of the text.

Of course the picture is much more complicated for general finite groups G. We can however say at least that over any algebraically closed field k of characteristic not dividing the order of G, this characteristic also fails to divide the order of any element of G, so that if π is a representation of G, the matrix $\pi(g)$ is diagonalizable for every $g \in G$ with eigenvalues that are roots of 1 in $\mathbb{C}$.

Example

Now consider the next simplest case, where $G = C_2 \times C_2$ is the Klein four-group, and assume that k has characteristic 2. Here G is generated by commuting elements x, y of order 2. I define a representation W of G of *infinite* degree as follows. A basis of W is given by $\{w_i\} \cup \{v_i\}$, where in both cases the index i runs over $\mathbb{Z}$. The generators x and y act trivially on each w_i (that is, they fix it). Then $xv_i = v_i + w_{i-1}, yv_i = v_i + w_i$ for $i \in \mathbb{Z}$. A picture of this representation is given by

$$\begin{array}{ccccccccc} \dots & & & w_0 & & w_1 & & & \dots \\ \dots & & v_0 & & v_1 & & v_2 & & \dots \end{array}$$

Example

Here x acts on the v_i by moving to the left, while y acts on the v_i by moving to the right. You can check directly that the actions of x and y commute and that acting by x twice or by y twice is the identity. You will show in HW next week that W is indecomposable. Thus G admits indecomposable representations of infinite degree; likewise it admits indecomposable representations of arbitrarily large finite degree.