Lecture 10-23: Ext groups and module extensions

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I will show that Ext groups are well defined (independent of the choice of projective resolution) and derive some long exact sequences.

Recall the setup from last time: we have projective resolutions $\{P_i\}, \{P'_i\}$ of the R-modules A and A', respectively, together with a map $f : A \to A'$. Denote the boundary map from P_n to P_{n-1} by d_n and the map from P_0 to A by d_0; similarly denote the boundary map from P'_n to P'_{n+1} by d'_n and the map from P'_0 to A' by d'_0. Last time we constructed maps $f_n : P_n \to P'_n$ such that $d'_n f_n = f_{n-1} d_n$.

We first prove

Proposition (p. 782)

With notation as above, the induced maps $\phi_n : \mathrm{Ext}^n_R(A', D) \to \mathrm{Ext}^n_R(A, D)$ depend only on f, not on the choice of lifts f_n used to define these maps.

Proof.

For this it is enough to show that if f is the zero map, then the ϕ_n are all zero as well. If $f = 0$ then commutativity of the diagram shows that f_0 maps P_0 into the kernel of d'_0; since d'_1 maps P'_1 onto this kernel, projectivity shows that there is a map $s_0 : P_0 \to P'_1$ lifting f_0. Then $d'_1(f_1 - s_0 d_1) = d'_1 f_1 - f_0 d_1 = 0$, so that $f_1 - s_0 d_1$ maps P_1 into the kernel of d'_1, which is the same as the image of d'_2. By projectivity there is a map $s_1 : P_1 \to P'_2$ with $f_1 = d'_2 s_1 + s_0 d_1$. Continuing in this way, we define homomorphisms $s_n : P_n \to P'_{n+1}$ such that $f_n = d'_{n+1} s_n + s_{n-1} d_n$. □

Proof.

The collection of maps $\{s_n\}$ is called a chain homotopy between the chain homomorphism given by the f_n and the zero homomorphism. In the presence of such a chain homotopy relative to a map $f : \mathcal{C} \to \mathcal{C}'$ between chain complexes $\mathcal{C}, \mathcal{C}'$, one checks that the induced map $f : H_n \to H_n'$ from any homology group of $\mathcal{C}$ to the corresponding one for $\mathcal{C}'$ is 0. A similar result holds for cochain complexes. In the present situation, taking homomorphisms into D, we get the cochain complex used to compute the ϕ_n. Thus $\phi_n = 0$ for all n, as desired. □

Now I can finally prove the independence result promised last time.

Theorem (p. 782)

The groups $\mathrm{Ext}_R^n(A, D)$ depend only on A and D.

Indeed, given two projective resolutions $\{P_i\}, \{P_i'\}$ of A and lifts f_n of the identity map $f : A \to A' = A$, the resulting maps ϕ_n from $\mathrm{Ext}_R^n(A, D)$ (computed via the P_i) and $\mathrm{Ext}_R^n(A', D)$ (computed via the P_i') admit maps $\phi_n : \mathrm{Ext}_R^n(A', D) \to \mathrm{Ext}_R^n(A, D), \phi_n' : \mathrm{Ext}_R^n(A, D) \to \mathrm{Ext}_R^n(A', D)$ such that the composites $\phi_n'\phi_n, \phi_n\phi_n'$ are induced from identity maps, so are both the identity. Thus ϕ_n, ϕ_n' are inverse isomorphisms and the assertion follows.

Given a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules and projective resolutions of L and N, say by $\{P_i\}$ and $\{P'_i\}$, respectively, it is not difficult to construct a projective resolution of M by $\{Q_i = P_i \oplus P'_i\}$ (p. 783) such that the resolutions of L, M, N fit into a short exact sequence of chain complexes. Taking homomorphisms into another R-module D and the corresponding long exact sequence in cohomology, I get a sequence

$$0 \to \hom_R(D, L) \to \hom_R(D, M) \to \hom_R(D, N) \xrightarrow{\gamma_0} \mathrm{Ext}^1_R(D, L)$$

$$\to \mathrm{Ext}^1_R(D, M) \to \mathrm{Ext}^1_R(D, N) \to \cdots$$

called the long exact sequence for Ext. This sequence explains the terminology *higher derived functors* for the Ext groups.

In particular, it follows from this sequence and the trivial projective resolution $\cdots \to 0 \to P \to P \to 0$ of any projective module P that an R-module P is projective if and only if $\mathrm{Ext}^1_R(P, B) = 0$ for all R-modules B, or if and only if $\mathrm{Ext}^n_R(P, B) = 0$ for all R-modules B and all $n \geq 1$.

The notation Ext here stands for "extension". An extension of an R-module N by another one L is a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules. Given two such extensions, say

$$0 \to L \to M \to N \to 0$$

$$0 \to L \to M' \to N \to 0$$

one says that they are equivalent if there is an isomorphism $f : M \to M'$ making the above diagram commute when supplemented by the identity maps from L and N to themselves (p. 381).

For example, taking $R = \mathbb{Z}$ and letting p be a prime, there is just one equivalence class of extensions of $L = \mathbb{Z}_p$ by itself with middle term isomorphic to $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Here the automorphism group G of M is the group $GL_2(\mathbb{Z}_p)$ of all invertible 2×2 matrices over $\mathbb{Z}_p$. Given a nonzero homomorphism $\pi : \mathbb{Z}_p \to E$ and two surjective homomorphisms $\phi_1, \phi_2 : E \to \mathbb{Z}_p$ whose kernels are both the image I of π, there is $g \in G$ that is the identity on I such that $\phi_1 g = \phi_2$. On the other hand, there are $p - 1$ inequivalent extensions of L with middle term $M' = \mathbb{Z}_{p^2}$. Here the automorphism group of M' is $Z^*_{p^2}$, the multiplicative group of units in M'. The unique subgroup S of M' isomorphic to $\mathbb{Z}_p$ is the one generated by the coset of p. There are p automorphisms of M' fixing S pointwise but $p(p-1)$ surjective homomorphisms from M' to $\mathbb{Z}_p$, each with kernel S. Thus there are indeed $p - 1$ inequivalent extensions altogether.

Thus there are in all p inequivalent extensions of L by itself. It is no coincidence that $\mathrm{Ext}^1_{\mathbb{Z}}(L, L)$ is cyclic of order p (by a previous calculation), as it is known in general that there is a bijection between $\mathrm{Ext}^1_R(N, L)$ and equivalence classes of extensions of N by L (Theorem 12, p. 787). In this bijection the zero element of the Ext group corresponds to the split extension $L \oplus N$. There is also the structure of an additive group on equivalence classes of extensions corresponding to the additive group structure on Ext^1.

Finally, I note briefly that the tensor product functor $D \otimes_R -$ likewise admits higher derived functors: using a projective resolution of the left R-module B and tensoring its terms with a. right R-module D, one defines Tor groups $\mathrm{Tor}_n^R(D, B)$ (p. 788; note the change in notation, with the ring R now appearing as a superscript). These groups are 0 for all B and all $n > 0$ if and only if D is flat. Similarly, they are 0 for all D and all $n > 0$ if and only if B is flat. In general one has $\mathrm{Tor}_0^R(D, B) \cong D \otimes_R B$.