Lecture 10-21: Homological algebra

October 21, 2024

I turn now to the first section of Chapter 17 of Dummit and Foote, which continues the material in Chapter 10. You have seen that not all left modules M over ring R are projective, so that given a short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to \hom_R(M, A) \to \hom_R(M, B) \to \hom_R(M, C) \to 0$ need not be exact. I will show how to measure this failure of exactness in a precise way.

To do this I first need to define and study the general notions of chain and cochain complexes. A sequence $\mathcal{C}$ of abelian group homomorphisms $\cdots \to C_n \to \cdots \to C_0 \to 0$ with $d_n : C_n \to C_{n-1}$, is called a chain complex if the composition of any two successive maps is 0 (p. 777). If instead $\mathcal{C}$ takes the form $0 \to C^0 \to C^1 \to \cdots \to C^n \to \cdots$, with $d_n : C^{n-1} \to C^n$, then it is called a cochain complex if the composition of two successive maps is 0. If $\mathcal{C}$ is a chain complex then its nth homology group $H_n(\mathcal{C})$ is the quotient $\ker d_n/\text{im } d_{n+1}$, where im denotes the image of a map; if $\mathcal{C}$ is a cochain complex then its nth cohomology group $H^n(\mathcal{C})$ is the quotient $\ker d_{n+1}/\text{im } d_n$. A chain (resp. cochain) complex is exact if and only if its homology (resp. cohomology) groups are trivial.

For convenience I mostly work with cochain complexes (following the text). The maps d_n are called coboundary maps and their kernels are called cocycles (for chain complexes they are called boundary maps and their kernels are called cycles). Given two complexes $\mathcal{A} = \{A^n\}, \mathcal{B} = \{B^n\}$, a homomorphism of chain complexes $\alpha : \mathcal{A} \to \mathcal{B}$ is a set of homomorphisms $\alpha_n : A^n \to B^n$ commuting with the coboundary maps. It is easy to check that such a homomorphism induces group homomorphisms from $H^n(\mathcal{A})$ to $H^n(\mathcal{B})$ for $n \geq 0$. A short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of cochain complexes is a pair of homomorphisms $\alpha : \mathcal{A} \to \mathcal{B}, \beta : \mathcal{B} \to \mathcal{C}$ such that the maps $0 \to A^n \to B^n \to C^n \to 0$ from a short exact sequence for all n. Then we have

Theorem: Long Exact Sequence in Cohomology, p. 778

Given a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, there are maps $\delta_n : H^n(\mathcal{C}) \to H^{n+1}(\mathcal{A})$ such that the sequence $0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \to H^1(\mathcal{A}) \to \cdots$ is exact.

You will work out the details of the proof in homework for next week). The maps δ_n are called connecting homomorphisms. A consequence of this result is that if any two of the cochain complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are exact then so is the third.

Now I can address the problem of measuring non-projectivity precisely. Henceforth R will be a fixed ring, A a left R-module, and all maps will be R-module maps.

Definition: projective resolution (p. 779)

A projective resolution of A is an exact sequence $P_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ of R-modules with the P_i projective; we denote by d_n the map $P_n \to P_{n-1}$ and by ϵ the map $P_0 \to A$.

Note that every module has a projective resolution, since free modules are projective: just a choose a free module P_0 surjecting onto A, say with kernel K_0, then a free module P_1 surjecting onto K_0, say with kernel K_1, and so on. If A is itself projective, then one has the very simple resolution $0 \to A \to A \to 0$; if A has a finite projective resolution (with only finitely many nonzero terms), then A is not too far from bring projective.

Now let A, D be two R-modules and $\{P_i\}_{i\geq 0}$ a projective resolution of A. Applying the contravariant functor $\hom(\cdot, D)$ to each term of the resolution, we get a cochain complex $0 \to \hom_R(A, D) \to \hom(P_0, D) \to \hom_R(P_1, D) \to \cdots$; denote again by $\epsilon, d_1, d_2, \ldots,$ the maps induced by the boundary maps $\epsilon, d_1, d_2, \ldots$ sending P_0 to A, P_1 to P_0, P_2 to P_1, and so on, in the resolution.

Definition: Ext groups, p. 779

With notation as above, we define $\mathrm{Ext}^n_R(A, D)$ to be the quotient of the kernel of d_{n+1} by the image of d_n for $n \geq 1$, while $\mathrm{Ext}^0_R(A, D) = \ker d_1$. The Ext groups are the *higher derived functor* groups of the contravariant functor $\hom_R(-, D)$. If $R = \mathbb{Z}$, then $\mathrm{Ext}^n_{\mathbb{Z}}(A, D)$ is denoted simply $\mathrm{Ext}^n(A, D)$.

Note that we are dropping the zeroth term $\hom_R(A, D)$ from the cochain complex.

I will show next time that these groups do not depend on the choice of projective resolution of A. For now I just note that this is true of Ext^0: since $\hom(\cdot, D)$ is left exact, the sequence $0 \to \hom_R(A, D) \to \hom_R(P_0, D) \to \hom_R(P_1, D)$ is exact, whence the kernel of d_1 in this sequence equals the image of ϵ, which is just $\hom_R(A, D)$: $\mathrm{Ext}^0_R(A, D) = \hom_R(A, D)$.

Example

Take $R = \mathbb{Z}$, $A = \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, A homomorphism ϕ from $\mathbb{Z}_m$ into A is completely determined by $\phi(1)$, which must be an element of A with $m\phi(1) = 0$. Hence $\mathrm{Ext}^0_{\mathbb{Z}}(A, D) \cong {}_mD$, where ${}_mD$ denotes the elements of D sent to 0 by m. There is a projective resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$, where the map from $\mathbb{Z}$ to itself is multiplication by m and the map from $\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ is the canonical one. Taking homomorphisms into D and applying the above definition we get $\mathrm{Ext}^0_{\mathbb{Z}}(A, D) \cong {}_mD$, $\mathrm{Ext}^1_{\mathbb{Z}}(A, D) \cong D/mD$, $\mathrm{Ext}^n_{\mathbb{Z}}(A, D) = 0$ for $n \geq 2$.

I now head toward the proof that the Ext groups are independent of the choice of projective resolution. First I show

Proposition, p. 781

Let $f : A \to A'$ be an R-module map and let $\{P_n\}, \{P'_n\}$ be projective resolutions of A, A', having boundary maps $\epsilon, d_1, d_2, \ldots$ and $\epsilon', d'_1, d'_2, \ldots$, respectively. Then there are lifts $f_i : P_i \to P'_i$ (together with f) making the obvious diagram commute.

This follows at once from the lifting property of projective modules: the map $P'_0 \to A'$is surjective, so the composite map $P_0 \to A \to A'$ lifts to a map $f_0 : P_0 \to P'_0$. Then the map from P'_1 to the kernel of the map from P'_0 to A' is surjective, so the composite map $P_1 \to P_0 \to P'_0$, whose image lies in this kernel, lifts to a map $f_1 : P_1 \to P'_1$, and so on. Then we also get induced lifts $f, f_0, f_1, \ldots$ from the terms of the cochain complex $0 \to \hom_R(A', D) \to \hom_R(P_0, D) \to \cdots$ to those of the complex $0 \to \hom_R(A, D) \to \hom_R(P_0, D) \to \cdots$ and thereby induced maps ϕ_n from $\mathrm{Ext}^n_R(A', D)$ to $\mathrm{Ext}^n_R(A, D)$.

Next time I will show that the maps $\phi_n : \mathrm{Ext}^n_R(A', D) \to \mathrm{Ext}^n_R(A, D)$ depend only on f, not on the choice of lifts f_n made above.