Lecture 10-18: Homomorphisms and tensor products

October 18, 2024

After briefly wrapping up the material on injective modules, today I return (for the last time) to tensor products, investigating the relationship between them and homomorphisms and the extent to which they preserve exact sequences.

I showed last time that a $\mathbb{Z}$-module is injective if and only if it is divisible, except that I need to complete the proof of Baer's Criterion. If the criterion holds for some R-module Q and I am given an R-module map f from a submodule A of another R-module B into Q, I need to show that this extends to a map from B into Q. Last time I showed that given $b \in B, .b \notin A$, I could extend f to $A + Rb$. I now invoke Zorn's Lemma (see Appendix 2 in the text) to argue that there is a maximal submodule B' of B to which I can extend f, which by the first part of the proof must be all of B, as desired.

Next I show that there are enough injective modules to contain any $\mathbb{Z}$-module.

Corollary 37, p. 397

Any $\mathbb{Z}$-module M is contained in an injective $\mathbb{Z}$-module.

Proof.

We can write $M \cong F/K$, where F is a free module, say with basis (f_i). Letting Q be the $\mathbb{Q}$-vector space with basis (f_i), there is an obvious inclusion of F into Q and thus an inclusion of M into Q/K. As Q/K, like Q, is easily seen to be injective, the result follows. □

In particular, divisible $\mathbb{Z}$-modules need *not* be $\mathbb{Q}$-vector spaces (this is a common mistake), though certainly any $\mathbb{Q}$-vector space is a divisible $\mathbb{Z}$-module. In later homework you will show

Theorem 38, p. 398

Any module M over any ring R is contained in an injective R-module.

In a nutshell, given M, the injective module containing it is $M' = \hom_{\mathbb{Z}}(R, Q)$, where Q is an injective $\mathbb{Z}$-module containing M and we make M' into an R-module via $rf(s) = f(sr)$ for $r, s \in R, f \in M'$. Thus although we have not captured all injective R-modules, we know at least that there are a lot of them.

Now I turn to the relationship (promised earlier) between homomorphisms and tensor products.

Theorem 43 (adjoint associativity), p. 401

Let R and S be rings, let A be a right R-module, B an (R, S)-bimodule, and C a right S-module. Then there is an isomorphism of abelian groups

$$\hom_S(A \otimes_R B.C) \cong \hom_R(A. \hom_S(B, C))$$

Here both groups are homomorphisms of *right* modules; $\hom_S(B, C)$ is a right R-module via the action $r \cdot f(x) = f(rx)$ for $r \in R, x \in B$.

Proof.

Given a homomorphism $\phi : A \otimes_R B \to C$, for each fixed $a \in A$ define $\Phi(a) : B \to C$ via $\Phi(a)(b) = \phi(a \otimes b)$ It is easy to check that $\Phi(a)$ is a homomorphism of right S-modules and that the map Φ from A to $\hom_S(B, C)$ mapping a to $\Phi(a)$ is a homomorphism of right R-modules. Thus $f(\phi) = \Phi$ defines a group homomorphism from $\hom_S(A \otimes_R B, C)$ to $\hom_R(A, \hom_S(B, C))$. Conversely, suppose $\Phi : A \to \hom_S(B, C)$ is a homomorphism. The map from $A \times B$ to C sending (a, b) to $\Phi(a)(b)$ induces a homomorphism $\phi : A \otimes_R B \to C$, whence $g(\Phi) = \phi$ defines a group homomorphism inverse to f and yields the desired isomorphism. □

Given a short exact sequence $0 \to L \to M \to N \to 0$ of left R-modules an right R-module D, note first that one has a natural sequence $0 \to D \otimes_R L \to D \otimes_R M \to D \otimes_R N \to 0$; if $f : L \to M, g : M \to N$, then the map f$1 \otimes f : D \otimes_R L \to D \otimes_R M$ sends a tensor $d \otimes x$ to $d \otimes f(x)$, while the map $1 \otimes g : D \otimes_R M \to D \otimes_R N$ sends $d \otimes y$ to $d \otimes g(y)$. Then $1 \otimes g$ has all decomposable tensors in its image, so is surjective. Given $d \otimes n \in D \otimes_R N$, choose any $m \in M$ with $g(m) = n$ and set $\pi(d \otimes n) = d \otimes m \in (D \otimes_R M)/M'$, where M' is the image of $1 \otimes f$. The map π is then well-defined and a two-sided inverse to $1 \otimes g$, so that the kernel of $1 \otimes g$ coincides with the image of $1 \otimes f$. We summarize this situation by saying that the functor $D \otimes_R -$, sending an R-module M to $D \otimes_R M$, is *right exact* (Theorem 39, p. 399). On the other hand, the first map from $D \otimes_R L$ to $D \otimes_R M$ need not be injective.

For example, if $R = \mathbb{Z}, D = \mathbb{Z}_n, L = n\mathbb{Z}, M = \mathbb{Z}$, and f is the inclusion of L into M, then both $L \otimes_{\mathbb{Z}} D$ and $M \otimes_{\mathbb{Z}} D$ are cyclic groups of order n, generated by $n \otimes 1$ and $1 \otimes 1$, respectively; but the tensor $n \otimes 1$, regarded as an element of $\mathbb{Z} \otimes_{\mathbb{Z}} D$, is 0, since it equals $1 \otimes n = 0$. Thus the covariant functor $D \otimes_R -$ is not exact in general.

Definition, p. 400

The right R-module D is called *flat* if $D \otimes_R -$ is exact on left R-modules, or equivalently if for any injection $L \to M$ of left R-modules the induced map $D \otimes_R L \to D \otimes_R M$ is also an injection.

As with injective modules, there is no uniform characterization of flat R-modules in general, but one does have the following result.

Proposition; Corollary 42, p. 390

Any projective R-module is flat.

Proof.

If P is projective, so that for some Q the direct sum $F = P \oplus Q$ is free, then tensoring with F is exact since it amounts to just replacing a module with the direct sum of r copies of itself, where r is the rank of F. Also we have seen that tensor products commute with direct sums (Theorem 17, p. 373). Exactness follows since a restriction of an injective map is again injective. □

Example

There are flat nonprojective modules, even over principal ideal domains. As an example, take $R = \mathbb{Z}$. If M is an R-module, then I have previously observed that any element of $N = M \otimes_R \mathbb{Q}$ takes the form $m \otimes \frac{1}{c}$ for some $c \in Rc \neq 0$. In order to decide when this element is 0, i give an equivalent construction of N, which will turn out to have considerable importance in Math 506 next spring. Let S be the set of nonzero elements of R. Denote by $S^{-1}M$ the set of equivalence classes of formal fractions $\frac{m}{c}$ for $m \in M, c \in S$, under the relation $\frac{m}{c} \sim \frac{n}{d}$ if $e(dm - cn) = 0$ for some $e \in S$.

Example

One easily checks that this is indeed an equivalence relation; it is a weakened version of the usual cross-multiplication condition for two fractions to be equal. Make $S^{-1}M$ into an R-module (celled the localization of M at S via the rules $\frac{m}{c} + \frac{n}{d} = \frac{dm+cn}{cd}, z\frac{m}{c} = \frac{zm}{c}$; these rules are well defined on equivalence classes. One has $m = \frac{m}{1} = 0$ in $S^{-1}M$ if and only if $sm = 0$ for some $s \in S$. It is then easy to check that the map sending $m \otimes \frac{1}{c}$ to $\frac{m}{c}$ is an isomorphism from N onto $S^{-1}M$. It follows that if $f : M \to P$ is an injection of R-modules, then $m \otimes \frac{1}{c} = 0$ in $M \otimes_R \mathbb{Q}$ if and only if $f(m) \otimes \frac{1}{c} = 0$ in $P \otimes_R \mathbb{Q}$, or if and only if $sm = 0$ for some $s \in R$.

Thus, in summary (see pages 402-3):

- For a left R-module M, the functors $\hom_R(M, -)$ and $\hom_R(-, M)$ from left R-modules to abelian groups are left exact but not exact in general. The first of these functors is covariant; the second is contravariant.
- For a right R-module M, the functor $M \otimes_R -$ from left R-modules to abelian groups is right exact but not exact in general. This functor is covariant.
- A left R-module is projective if and only $\hom_R(M, -)$ is exact, or if and only if M is a direct summand of a free module.
- A left R-module is injective if and only if $\hom_R(-, M)$ is exact. Over principal ideal domains R, M is injective if and only if it is divisible, or $M = rM$ for all $r \in R, r \neq 0$.
- A right R-module M is flat if and only if the functor $M \otimes_R -$ is exact. Projective modules are flat.