Lecture 10-11: Tensor products I

October 11, 2024

Modules over rings do not have product structures on them, but given two modules over the same ring, one can form a third module which can be viewed as the product of the first two. The construction is very general and difficult to grasp all at once, so I will introduce it in stages, making different hypotheses on the base ring at each stage.

Assume to begin with that R is commutative and let M and N be two R-modules. In order to define the product of these modules, one would have to make sense of the expression $m \times n$ for $m \in M, n \in N$. If one does this in the most general possible way, assuming only those properties of $m \times n$ that would hold for any product worthy of the name, one is led to

Definition; see equation 10.6 on p. 364

The *tensor product* $M \otimes_R N$ *of* M *and* N *over* R is the quotient of the free R-module F on the Cartesian product $M \times N$ by the submodule S generated by

- $((m_1 + m_2), n) - (m_1, n) - (m_2, n)$ for $m_1, m_2 \in M, n \in N$;
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$ for $m \in M, n_1, n_2 \in N$; and
- $r(m, n) - (rm, n), r(m, n) - (m, rn)$ for $r \in R, m \in M, n \in N$.

The image of (m, n) in $M \otimes_R N$ is called the tensor product of m and n and is denoted $m \otimes n$. It is called a *(decomposable) tensor*.

Thus an element of $M \otimes_R N$ is a finite sum of tensors (but not necessarily a single tensor). The third set of generators $r(m,n) - (rm,n), r(m,n) - (m,rn)$ of S demonstrates the role that the subscript R plays in the definition of $M \otimes_R N$. An expression like $m \otimes n$ is unambiguous only if we know exactly which scalars we can move past the tensor product sign. Note that every bilinear or balanced map from the Cartesian product $M \times N$ to an R-module P, that is, every map f satisfying $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n), f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2), f(rm, n) = f(m, rn) = rf(m, n)$, corresponds uniquely to an R-module map $g : M \otimes_R N \to P$ (Theorem 10, p. 365).

One has the commutative law $M \otimes_R N \cong N \otimes_R M$ (Proposition 21, p. 374), the associative law $(M \otimes_R N) \otimes_L \cong M \otimes_R (N \otimes_R L)$ (Theorem 14, p. 371), and the distributive law $M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$ (Theorem 17, p. 373).

Example

The simplest case of this construction occurs when both M and N are free over R, say with bases $\{b_1, \ldots, b_m\}$ and $\{c_1, \ldots, c_n\}$, respectively. In this case a typical tensor $t = (\sum_i r_i b_i) \otimes (\sum_j s_j c_j)$ can be rewritten as $\sum_{ij} r_i s_j (b_i \otimes c_j)$. The map sending t to $s = \sum_{ij} r_i s_j b_i c_j$, regarded as an element of the free module P on the $b_i c_j$, extends to an R-module homomorphism from F to P which is easily seen to send the submodule S to 0 and to define an isomorphism from $F/S = M \otimes_R N$ to P. Thus the tensor product of two free modules over R is another free module whose rank is the product of the ranks of the factors. This is what we would expect, given the product terminology. But notice that the subscript is crucial: the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is free of rank $2 \cdot 2 = 4$ over $\mathbb{R}$, while the tensor product $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$) is isomorphic to $\mathbb{C}$ and so free of rank 2 over $\mathbb{R}$ (Example 4, p. 375).

Example

Matters get a little more complicated once one moves beyond free modules. For example, the tensor product $T = (R/I) \otimes_R (R/J)$ of two cyclic modules is the cyclic module $R/(I+J)$, where I, J are ideals of R. To see this, observe that T is clearly cyclic, being generated by $1 \otimes 1$, and that $i(1 \otimes 1) = j(1 \otimes 1) = 0 \in T$ for all $i \in I, j \in J$. Then the map sending $\sum_i \bar{r}_i \otimes \bar{s}_i \in T$ to $\sum_i \overline{r_i s_i} \in R/(I+J)$, where $\bar{r}_i, \bar{s}_i$ denote the respective cosets of r_i, s_i in $R/I, R/J$ is an isomorphism. Note that I exploited the product structure in R and its quotients $R/I, R/J$ to understand the structure of T. In particular, if the sum $I+J$ is all of R, then $(R/I) \otimes_R (R/J) = 0$ even though neither factor is 0.

Example

The tensor product of a free and a nonfree module can also be smaller than one might expect. For example, we have $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, since given a typical tensor $x \otimes y$ we can write $y = nz$ for some $z \in \mathbb{Q}$ and then $x \otimes y = nx \otimes z = 0$. This vanishing property will come in handy when I prove the uniqueness part of the classification of finitely generated modules over a principal ideal domain. We also have $T = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, since a tensor $\frac{a}{b} \otimes \frac{c}{d}$ can be rewritten first as $\frac{ac}{b} \otimes \frac{1}{d}$ and then as $ac \otimes \frac{1}{bd}$. Thus $T \cong T' = \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$; by analogy with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ one could have expected T to be larger than T'.

I now drop the assumption that R is commutative. Let R' be another ring and let M be an $R' - R$ bimodule, so that M is simultaneously a left R'-module and a right R-module and $(r'm)r = r'(mr)$ for $r \in R, r' \in R'$, and $m \in M$ (Definition, p. 366). Taking N to be a left R-module, define the tensor product $M \otimes_R N$ to be the quotient F/S defined above, except that the third set of generators for the submodule S should now be $(mr, n) - (m, rn)$; note that this choice of generator is more natural than the earlier one if R is not commutative. Then $M \otimes_R N$ is no longer either a left or right R-module, but it is a left R'-module via the recipe $r'(m \otimes n) = r'm \otimes n$ for $r' \in R', m \in M, n \in N$. (One easily checks that this action is compatible with the generators used to define $M \otimes_R N$.) This construction applies in particular if $M = R'$, R is a subring of R', and N is a left R-module; one says in this case that $R' \otimes_R N$ is obtained from N by extending scalars or by change of ring.

Finally, in the most general setting, M is a right R-module and N a left R-module. Defining $M \otimes_R N$ as above, one finds that it is still an abelian group, but now lacks any R-module structure. For example, if $M = R/I, N = R/J$ with I a right ideal and J a left one of R, then $M \otimes_R N \cong R/(I + J)$; even though $I + J$ is neither a left nor a right ideal of R, it is still an additive subgroup, so that the quotient by $I + J$ makes sense. Recall that I earlier observed that the space $\hom_R(M, N)$ of R-module homomorphisms between a pair M, N of left R-modules is likewise an abelian group but not an R-module (unless R is commutative). In fact, the tensor product and homomorphism constructions are closely related; I will say more about both of these constructions later.

For now, I will close by returning to a special case of the first situation I considered today. Assume once again that R is commutative and let V be a free R-module of finite rank n. One can construct the tensor product $T_m = T^m V = \otimes_R^m V$ of any number of copies of V, defined in the (essentially) obvious way; this is a free R-module of rank n^m, called the mth tensor power of V. The quotient $T^m V/E$ of T_m by the submodule E generated by all tensors of the form $\ldots \otimes v \otimes \ldots \otimes v \otimes \ldots$ as v runs over V called the mth exterior power of V and denoted $\bigwedge_m = \bigwedge^m V$ (see the Definition on p. 446). Decomposable elements of it are denoted $v_1 \wedge \ldots \wedge v_m$ with the $v_i \in V$. As a consequence of its definition the fundamental relation $(\ldots \wedge v \wedge \ldots \wedge w \wedge \ldots) = -(\ldots \wedge w \ldots \wedge v \wedge \ldots)$ holds in $\bigwedge_m$ for all $v, w \in V$. $\bigwedge_m$ is free over R with basis consisting of all $v_{i_1} \wedge \ldots v_{i_m}$, where $v_1, \ldots, v_n$ is a basis of V and the indices i_j satisfy $i_1 < i_2 < \ldots$.

The rank of $\bigwedge_m$ is thus the binomial coefficient $\binom{n}{m}$; this rank is 0 whenever $m > n$, so that $\bigwedge_m = 0$ in that case. One also has the mth symmetric power $S_m = S^m V$, defined to be the quotient T_m/S, where S is the submodule generated by all tensors of the form $\ldots \otimes v \otimes \ldots \otimes w \otimes \ldots$ as v, w range over V (see the Definition on p. 444). Decomposable elements of S_m are denoted $w_1 \otimes \ldots \otimes w_m$ or just $w_1 \ldots w_m$ where the w_i lie in V; here we have $\ldots w_i \ldots w_j \ldots = \ldots w_j \ldots w_i \ldots$. Like $\bigwedge_m$, S_m is free over R. A basis of it consists of all $v_{i_1} \ldots v_{i_m}$ as the v_{i_j} run through a basis of V, where this time the indices satisfy $i_1 \leq i_2 \leq \ldots$. The rank of S_m turns out to be $\binom{n+m-1}{m}$. Thus we never have $S_m = 0$, unlike the situation for $\bigwedge_m$, and in fact the rank of $S_m V$ increases as m does.

The direct sum $T = T(V) = \oplus_{i=0}^{\infty} T_i$ of the T_i (taking $T_0 = R, T_1 = V$) then has a ring structure; here one decrees that $(v_1 \otimes \ldots \otimes v_m)(w_1 \otimes \ldots \otimes w_r) = v_1 \otimes \ldots \otimes w_r$ (and extends to arbitrary products by the distributive law). T is called the tensor algebra of V (Definition, p. 443). It is a graded ring: we have $T_i T_j \subset T_{i+j}$ for all indices i, j. The product structure extends to $S = S(V) = \oplus_{i=0}^{\infty} S^i V$ and $\bigwedge^k = \bigwedge^k V = \oplus_{i=0}^{\infty} \wedge^i V$ in the obvious way; these are called the symmetric and exterior algebras of V. These too are graded rings. Note that the exterior algebra of V has finite rank 2^n as an R-module, while the symmetric algebra has infinite rank.

In practice tensor, symmetric, and exterior algebras arise most commonly in the context of a finite-dimensional vector space V over a field K. The one-dimensionality of $\bigwedge^n V$ plays a crucial role in the development of the determinant in Section 11.4 of the text.