MIDTERM #2 SOLUTIONS

1. State the basic properties proved in class about continuous functions on a closed bounded interval and indicate which of these also hold for derivatives on such an interval.

The properties I had in mind are the Intermediate Value Property (that the function f takes on all values between $f(a)$ and $f(b)$ in the interval is $[a, b]$) and the Extreme Value Property (that f has a maximum and minimum on $[a, b]$). Derivatives satisfy the first of these but not the second. A number of people mentioned uniform continuity, but since derivatives need not even be continuous, they certainly need not be uniformly continuous.

2. Let a_k be a periodic sequence of signs, so that there is an integer $N > 0$ with $a_{k+2N} = a_k$ for all k and each $a_k = \pm 1$. Assume also that $\sum_{k=1}^{2N} a_k = 0$ and let b_k be a sequence of real numbers such that $b_k \geq b_{k+1}$ for all k and $b_k \to 0$ as $k \to \infty$. Use a theorem in class to show that the *series* $\sum_{k=1}^{\infty} a_k b_k$ converges.

This follows from Dirichlet's Test, but for full credit you needed to verify that the partial sums of $\sum a_k$ are bounded. This requires both of the additional hypotheses that the a_k are periodic and $\sum_{k=1}^{2N} a_k = 0$; given these, it follows that the partial sums of $\sum a_k$ are likewise periodic, so that only finitely many partial sums occur and it is clear that they are bounded. It is not enough to observe that the a_k themselves are bounded; one must also look at partial sums of them.

3. Work out a power series expansion of $g(x) = e^{x^3}$, by starting with a power series for e^x and then making a suitable change of variable.

Few people had any trouble with this one. Starting with the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for e^x, replace x by x^3 to get $\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

4. Correct the following *misstatements* of theorems proved in class (you need not prove the corrected versions).

(a) If the pointwise limit f of a sequence of continuous functions f_n is continuous, then the convergence is uniform.

(b) If f is a continuous function on $\mathbb{R}$ and $C \subset \mathbb{R}$ is connected, then the inverse image $f^{-1}(C)$ is connected.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable (i.e. has derivatives of all orders), then f has a Taylor expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with a positive radius of convergence.

This problem caused the most trouble. In part (a) you had to reverse the hypothesis and conclusion: if the convergence is uniform, then the limit f is continuous. In part (b), you had to replace the inverse image $f^{-1}(C)$ by the image $f(C)$. In part (c) you again had to reverse hypothesis and conclusion: if f has a Taylor expansion with positive radius of convergence, then it is infinitely differentiable.

5. Give an example (possibly using a theorem in class) of a function f that is the uniform limit of differentiable functions on an interval $[a, b]$ but is not differentiable on that interval.

By a general result in class (the Weierstrass Approximation Theorem), any continuous function on $[a, b]$ is the uniform limit of polynomials, but since by an example in class there are continuous functions on this interval that are nowhere differentiable, any such function provides an example. Alternatively, I showed directly in class (while proving the Weierstrass Theorem) that $f(x) = |x|$ is the uniform limit of polynomials on say $[-1, 1]$; but this function fails to be differentiable at $x = 0$.