MIDTERM #1 SOLUTIONS

1. Show carefully, using the definition of limit, that $\frac{4}{7n^2+3} \to 0$ as $n \to \infty$.

We have $0 < \frac{4}{7n^2+3} < \frac{4}{7n^2}$ for all n. Given $\epsilon > 0$, choose any integer N with $N^2 > \frac{4}{7\epsilon}$; then if $n > N$ we have $\frac{4}{7n^2} < \epsilon$, whence $|\frac{4}{7n^2+3}| < \epsilon$, as desired. Most people got this one; some got tangled up in the algebra to find the proper N.

2. The following are *incorrect* versions of theorems proved in class. In each case give the *correct* statement of the theorem.

(a) A bounded sequence converges if and only if it is monotone.

A monotone sequence converges if and only if it is bounded.

(b) Every nonempty set of real numbers that is bounded below has a least lower bound.

Every nonempty set of real numbers bounded below has a greatest lower bound.

(c) There are no subsets of $\mathbb{R}$ that are both closed and open.

The only subsets of $\mathbb{R}$ that are both closed and open ore $\emptyset$ and $\mathbb{R}$ (or there are exactly two subsets of $\mathbb{R}$ that are both open and closed).

3. Let S be a nonempty set of real numbers that is bounded below and $\{C_x : x \in S\}$ be the corresponding set of cuts of rational numbers. Work out an expression for C, the cut corresponding to the greatest lower bound of S, in terms of the C_x.

Take the intersection of all the C_x with $x \in S$, with the largest element removed if there is one, to get the greatest lower bound. This question was the biggest source of lost points for most students; you had to give an explicit construction of the greatest lower bound, not invoking anything about the real numbers to do this. Note that the parallel construction of the least upper bound in class was given by the union of the C_x.

4. Show that $\sum_{i=1}^{\infty} \frac{\sin^2 i}{2^i}$ converges, by showing that its partial sums are bounded.

A typical partial sum $\frac{\sum_{i=1}^{n}(\sin^2 i}{2^i}$ is (positive and) bounded above by $\sum_{i=1}^{n}(1/2^i)$; such sums are in turn bounded by 1, by the formula for the sum of a geometric series (this is not, by the way, a p-series).

5. Decide whether the series $\sum_{n=1}^{\infty} \frac{5n+4}{n^3+7}$, converges, making sure to state explicitly which test you are using.

Taking the limit of the ratio $\frac{(5n+4)/(n^3+7)}{5/n^2}$ as $n \to \infty$, we get the finite nonzero number 5, whence the given series and $\sum_{n=1}^{\infty} \frac{5}{n^2}$ converge or diverge together, by the Limit Comparison Test. But the latter series is a multiple of the p-series with $p = 2$, so it converges.