FINAL EXAM SOLUTIONS

1. Show that there is no continuous function g on the interval $[0, 1]$ such that $g(\frac{1}{2n}) = \frac{1}{3}$ while $g(\frac{1}{2n+1}) = \frac{1}{2}$ for all positive integers n.

Any such function would have to take the contradictory values $\frac{1}{3}$ and $\frac{1}{2}$ at 0, we must have $g(0) = \lim g(s_n)$ for any sequence s_n converging to 0.

2. Write down a cut C of rational numbers whose square is 2. Be sure to include only rational numbers in C and to describe it purely in terms of rational numbers.

Take $C = \{x \in \mathbb{Q}, x < 0\} \cup \{xin\mathbb{Q} : x^2 < 2\}$. 3. Work out a power series expansion of $\int_0^x \sin(t^2)\, dt$. For which x does this series converge?

Changing variables and integrating, we get $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$. The series converges for all x.

4. Let f be a continuous real-valued function defined on the Cantor set C. Show that f has a maximum and a minimum on C.

this follows at once since C is both closed and bounded in $\mathbb{R}$.

5. The following statements are either theorems proved in class, or else *misstatements* of such theorems. Correct each, or mark it as already correct.

(a) If the derivative of a function f is nonnegative at a point c, then f is weakly increasing in some interval $[a, b]$ containing c.

If f is weakly increasing in an interval $[a, b]$ containing c, then $f'(c) \geq 0$.

(b) If $\sum |a_n|$ converges, then so does $\sum a_n$.

Correct.

(c) The series $\sum_{n=1}^{\infty} x^{-n}$ converges for $|x| < 1$.

This series converges for $|x| > 1$.

6. Determine the set of positive real numbers k such that $\sum_{n=0}^{\infty} \frac{1}{1+n^k}$ converges. Indicate which tests you are using to determine this set.

Combining the limit comparison and p-series test, we get that the series converges exactly for $k > 1$.

7. Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} e^{\frac{i+\frac{1}{2}}{n}}$, by realizing this sum as a Riemann sum of a suitable function.

This is a Riemann sum for $\int_0^1 e^x \, dx$, so the limit is $e - 1$.

8. Show that there is $x \in \mathbb{R}$ with $\cos x = x$.

This follows at once from the Intermediate Value Theorem applied to $f(x) = \cos x - x$, since $f(0) > 0$ and $f(1) < 0$.