# Lecture 6-6: Review

June 6, 2025

In this lecture I briefly review once again the convergence theorems for sequences and series, uniform convergence, and Taylor series.

I will skip the convergence tests for series, since these were reviewed before both midterms. Recall that a sequence $f_n$ of functions converges uniformly to its limit $f$ on a set $S$ if for every $\epsilon > 0$ there is an index $N$ such that $|f(x) - f_n(x)| < \epsilon$ for all $n > N$ and $x \in S$; given $\epsilon$, the same $N$ must work for all $x$ simultaneously. We say that $f$ is the uniform limit of the $f_n$ in this case. Then the uniform limit of continuous functions is continuous (but not in general the pointwise limit); similarly the uniform limit on an interval $[a, b]$ of integrable functions $f_n$ is integrable. The same is *not* true of differentiable functions $f_n$; even the uniform limit of differentiable $f_n$ can fail to have a derivative at any point (though it must still be continuous).

A handy criterion for *series* $\sum f_n$ of functions on a set $S$ to converge uniformly is the Weierstrass $M$-test: if there is a convergent series $\sum M_i$ of nonnegative numbers $M_i$ such that $|f_n(x)| \leq M_n$ for $x \in S$ and indices $n$, then $\sum f_n$ converges uniformly on $S$, so that in particular the sum is continuous (or integrable) if the $f_n$ are.

The most important class of uniformly convergent series of functions is that of Taylor series of analytic functions. Given an infinitely differentiable function $f$ on an open interval containing a point $a$, its Taylor series at $x = a$ is the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. Like any Taylor series $\sum_{n=0}^{\infty} a_n(x-a)^n$, this series has a radius of convergence $R$, so that it converges if $|x-a| < R$ and diverges if $|x-a| > R$, with either behavior possible if $|x-a| = R$. A formula for $R$ is given in general by $\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$ whenever the limit exists. In general, there is no guarantee that the Taylor series of a $f$ has a positive radius of convergence, or even if it does that the series converges to $f$. What *is* guaranteed is that if a function $f(x)$ admits a Taylor series expansion $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging to $f$ on some open interval $(a-R, a+R)$, then $f$ is infinitely differentiable on this interval and we must have $a_n = \frac{f^{(n)}(a)}{n!}$ for all $n$. Moreover, if this series has radius of convergence $R > 0$ and converges to $a+R$, it must converge to the left-hand-limit of $f$ at that point; similarly for $a-R$.

I used differential equations to show that certain functions are indeed analytic at $x = a$ for suitable $a$, that is, the sum of their Taylor series at that point. For example, one has $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$. For $|x| < 1$ one also has $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$, $(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n$ for any constant $\alpha$. The series for $-\ln(1-x)$ converges also at $x = -1$ to $-\ln 2$; the series for $(1+x)^\alpha$ converges at $x = 1$ to $2^\alpha$, for any $\alpha$, and at $x = -1$ to 0 for any $\alpha > 0$.

Often we can change variables to derive new Taylor series from old ones. Thus given the series for $\sin x$, namely $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, we get the series for $\sin x^3$ by simply replacing $x$ throughout by $x^3$ in the above series. We get $\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!}$. Integrating term by term, we get the series for $\int_0^x \sin t^3 \, dt$, even though there is no formula for this last function. This series is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+4}}{(6n+4)(2n+1)!}$ All of these series converge absolutely for all $x$.

Now I review the definition of the integral of a function. Let $f$ be a function bounded on a closed interval $[a, b]$. For every partition $P$ of $[a, b]$, that is, for every finite set of points $\{x_0, \ldots, x_n\}$ with $x_0 = a < x_1 < \ldots < x_n = b$, define the upper sum $(U(f, P)$ and lower sum $L(f, P)$ by $\sum_{i=0}^{n-1} M_i(x_{i+1} - x_i), \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$, respectively, where $M_i$ is the supremum of $f$ on $[x_{i-1}, x_i]$ and $m_i$ is the infimum of $f$ on the same interval. Then we have $L(f, P) \leq U(f, Q)$ for all partitions $P, Q$. If there is a unique number $I$ such that $L(f, P) \leq I \leq U(f, P)$ for all partitions $P$, then we say that $f$ is integrable on $[a, b]$ and write $I = \int_a^b f(x)\, dx$. In general, we write $\underline{\int}_a^b f(x), dx, \overline{\int}_a^b f(x)\, dx$, respectively, for the supremum of all $L(f, P)$ and the infimum of all $U(f, P)$, calling these numbers the lower and upper integrals of $f$ over $[a, b]$.

If $f$ is continuous on $[a, b]$, or more generally even if $f$ is just assumed to be integrable on $[a, b]$, then we have $I = \int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(y_i)$, where $y_i$ is any point in the closed interval $[a + i(\frac{b-a}{n}), a + (i+1)(\frac{b-a}{n})]$. Given any $\epsilon > 0$, for all sufficiently large $n$ the given sum (called a Riemann sum for $f$) is guaranteed to fall within $\epsilon$ of $\int_a^b f(x)\,dx$, for any choice of $y_i$. For particular computations the choice $y_i = a + i(\frac{b-a}{n})$, or $y_i = a + \frac{i+1/2}{n}(b-a)$, may be especially convenient. Given a limit of sums that can be interpreted as a Riemann sum of a suitable function, you should be able to recognize this function and use the Fundamental Theorem of Calculus to evaluate its integral, thereby computing the original limit.

The strengthened form of the Fundamental Theorem of Calculus that you learned this term is that if $f$ is integrable on $[a, b]$ and continuous at $x \in [a, b]$, then $f$ is also integrable on $[a, x]$ and if we set $F(y) = \int_a^y f(t)\,dt$ for all $y \in [a, b]$, then $F$ is differentiable at $x$ and $F'(x) = f(x)$. Even more generally, if you assume only that $f$ is integrable on $[a, b]$, then the function $F(y)$ is continuous.

Finally, a reminder that the final exam is scheduled for Monday, June 9, at 2:30, in the usual classroom Mueller 155. Logistics are the same as for the midterms: you are allowed both sides of one sheet of handwritten notes and all work is done on the test paper. Good luck and have a good summer!