Lecture 5-9: Series of functions

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I now return to sequences and series, this time looking at sequences and series of functions rather than numbers, following Chapter 6 of the text. Let $(f_n(x))$ be a sequence of functions f_n defined on an interval I such that the sequence $f_n(x)$ of numbers converges, say to $f(x)$, for all $x \in I$. The function f is then called the pointwise limit of the f_i. See Definition 6.2.1 on p. 174.

Example

If $f_n(x) = x^n$ and $I = [0, 1]$, then $f(x) = 0$ for $0 \leq x < 1$, since $\lim_{n\to\infty} x^n = 0$, while $f(1) = 1$. See Example 6.2.2 (ii), p. 174. Observe that each $f_n(x)$ is continuous on I but $f(x)$ is not. Note also that the derivative g' of any differentiable function g is the pointwise limit of the continuous functions $g_n(x) = \frac{g(x+\frac{1}{n})-g(x)}{\frac{1}{n}}$.

This example shows that pointwise limits of sequences of functions can fail to have the same nice properties as the functions themselves. One would like conditions under which this kind of pathology does not occur, so that the limit of continuous functions is continuous. Note first that it is not a question here of being defined on a closed bounded interval, since in the example I is closed and bounded.

Instead I need to impose conditions on the convergence here. I say that the functions $f_n(x)$ converge to $f(x)$ uniformly and call $f(x)$ the uniform limit of the $f_n(x)$ if for every $\epsilon > 0$ there is an index N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and *all* $x \in I$ (see Definition 6.2.3, p. 177). This condition depends on both the functions $f_n(x)$ and the interval I; observe for example that the convergence of $f_n(x)$ to $f(x)$ is uniform on $[0, \alpha]$ for any $\alpha \in (0, 1)$: on this interval the function $f(x)$ is the 0 function, and the increasingness of $f_n(x)$ shows that if N is chosen so that $\alpha^N < \epsilon$, then indeed $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in [0, \alpha]$ (since also $\alpha^n \leq \alpha^N$ in this situation).

On the other hand, the convergence of the same sequence $f_n(x)$ of functions to $f(x)$ is *not* uniform on $(\beta, 1)$ for any $\beta < 1$. If it were, then for $\epsilon = 1/2$ I would have in particular $|f_N(x) - f(x)| = x^N < 1/2$ for $x \in (\beta, 1)$. Letting x approach 1 from below, I get a contradiction, since $\lim_{x \to 1^-} x^N = 1$. The theorem that emerges from this example and this definition is then

Continuous Limit Theorem (6.2.6, p. 178)

If $f(x)$ is the uniform limit of continuous functions $f_n(x)$ on an interval I, then $f(x)$ is continuous on I.

Proof.

Now I get to do an $\epsilon/3$ proof, for the first time in the course. Given $x \in I$ and $\epsilon > 0$, first choose N so that $|f(x) - f_n(x)| < \epsilon/3$ for $n \geq N$ and $x \in I$. Then choose $\delta > 0$ so that $|f_N(x) - f_N(y)| < \epsilon/3$ whenever $|x - y| < \delta$ and $y \in I$. Then I get $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{3\epsilon}{3} = \epsilon$, as desired, under the same hypothesis on x and y. □

I define an infinite series $\sum_{i=1}^{\infty} f_i(x)$ of functions in the same way as for infinite series of numbers, namely as the corresponding sequence $s_n(x) = \sum_{i=1}^{n} f_i(x)$ of partial sums of the functions. I say that $\sum_{i=1}^{\infty} f(x)$ converges to $f(x)$ uniformly on an interval I if $s_n(x)$ converges to $f(x)$ uniformly on I. This occurs if and only if the sequence of functions $t_n(x)$ converges to 0 uniformly on I, where $t_n(x) = \sum_{i=n}^{\infty} f_i(x)$. There is a similar definition for $\sum_{i=1}^{\infty} f_i(x)$ to be uniformly Cauchy on I.

Weierstrass M-test (Corollary 6.4.5, p. 189)

Given a series $\sum_{i=1}^{\infty} f_i(x)$ of functions $f_i(x)$ on an interval I, suppose there is a convergent series $\sum_{i=1}^{\infty} M_i$ of numbers M_i such that $|f_i(x)| < M_i$ on I. Then $\sum_{i=1}^{\infty} f_i(x)$ converges absolutely and uniformly to its sum on I. In particular, this sum is continuous on I if the f_i are.

Proof.

The partial sums of $\sum_{i=1}^{\infty} M_i$ are bounded, whence the partial sums of $\sum_{i=1}^{\infty} |f_i(x)|$ have the same bound and $\sum_{i=1}^{\infty} f_i(x)$ converges absolutely. Denote its sum by $f(x)$. The difference $f(x) - \sum_{i=1}^{n} f_i(x) = \sum_{i=n+1}^{\infty} f_i(x)$ is then uniformly bounded by a constant approaching 0 as n goes to infinity and $\sum_{n=1}^{\infty} f_i(x)$ converges uniformly, as desired. $\square$

Another example of the benefits of uniform convergence comes from integration. Here, in addition to the Fundamental Theorem of Calculus mentioned previously, I use that $\int_a^b f(x)\,dx$ exists if f is continuous on $[a, b]$; that this integral can be interpreted as the area under the graph of $f(x)$ on this interval, assuming that $f(x)$ is nonnegative there; that integration is linear, taking sums of functions to the sum of their integrals; and that $|\int_a^b f(x)\,dx)| \leq \int_a^b |f(x)|\,dx$. Also if $0 \leq f(x) \leq g(x)$ on $[a, b]$ and $f(x), g(x)$ are integrable on this interval, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Example

If $f_n(x) = n^2x$ for $x \in [0, 1/n]$, $f_n(x) = 2n - n^2x$ for $x \in [1/n, 2/n]$, $f_n(x) = 0$ for all other $x \in [0, 2]$, then $f_n(x) \to f(x) = 0$ pointwise on $[0, 2]$, since for any $x > 0$ we have $f_n(x) = 0$ for all sufficiently large n, while $f_n(0) = 0$ for all n. Yet $\int_0^2 f_n(x)\, dx = 1$ for all n, so that $\int_a^b f(x)\, dx$ is not the limit of $\int_a^b f_n(x)\, dx$.

Once again uniform convergence saves the situation.

Theorem 7.4.4, p. 231

If $f_n(x)$ is a sequence of continuous functions converging uniformly to $f(x)$ on an interval $[a, b]$, then $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.

Proof.

Since the $f_n(x)$ converge uniformly to $f(x)$, it follows that $f(x)$ is continuous and thus integrable on $[a, b]$. Given $\epsilon > 0$, choose N so that $|f(x) - f_n(x)| < \frac{\epsilon}{b-a}$ on $[a, b]$ for $n \geq N$. Then $|\int_a^b f_n(x) - f(x)\,dx| \leq \int_a^b |f_n(x) - f(x)|\,dx < \epsilon$ for $n \geq N$, as desired. □

By contrast, given a uniformly convergent sequence (f_n) of differentiable functions f_n on an interval I, there is no guarantee that the limit function f is even differentiable, much less that the sequence $f_n'(x)$ converges to $f'(x)$ for any $x \in I$. Using uniform convergence I will develop the theory of Taylor series next week; that is, of series of functions of the form $\sum_{k=0}^{\infty} a_k(x-a)^k$ for $a_k \in \mathbb{R}$ and a fixed constant a.