Lecture 5-5: Derivatives, continued

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Continuing with derivatives, we now show that inverses of one-to-one differentiable functions are themselves differentiable, provided that their derivatives are never 0.

Theorem; cf. Exercise 5.2.12, p. 155

Let f be differentiable and one-to-one on the interval $[a, b]$ and suppose that $x_0 \in (a, b)$ is such that $f'(x_0) \neq 0$. Set $y_0 = f(x_0)$. Then the inverse function $g = f^{-1}$, defined on the interval $[c, d] = f([a, b])$ is differentiable at y_0 with $g'(y_0) = \frac{1}{f'(x_0)}$.

Proof.

We have already seen that $[c, d] = f([a, b])$ is indeed a closed interval. Then we have

$1 = \lim_{y \to y_0} \frac{f(g(y)) - f(g(y_0))}{y - y_0} = \lim_{y \to y_0} \frac{f(g(y)) - f(g(y_0))}{g(y) - g(y_0)} \frac{g(y) - g(y_0)}{y - y_0}$.

Moreover we cannot have $g(y) = g(y_0)$ for values of y arbitrarily close to but unequal to y_0, since then the quotient $\frac{f(g(y)) - f(g(y_0))}{y - y_0}$ as y approaches y_0 through a sequence of such values would approach 0 rather than 1. Then continuity of g at y_0 forces the first fraction to approach $f'(g(y_0) = f'(x_0) \neq 0$ as $y \to y_0$, whence the second fraction approaches $\frac{1}{f'(x_0)} = g'(y_0)$ as $y \to y_0$, as claimed. □

The same argument shows that g is *not* differentiable at y_0 if $f'(x_0) = 0$, since if it were we would have $1 = 0g'(y_0)$.

Example

If $f(x) = x^n$ with $n \in \mathbb{N}$, then we have already seen that $f'(x) = nx^{n-1}$ for all x; we also know that $f(x)$ has a continuous inverse $g(x) = x^{1/n}$ defined for $x \geq 0$. Evaluating $f'(g(x))$ we get $nx^{\frac{n-1}{n}}$ if $x > 0$; taking the reciprocal we get $g'(x) = \frac{1}{n}x^{\frac{1-n}{n}}$ for $x > 0$, but $g'(0)$ does not exist, as predicted by the above proof.

Example

Before giving a second example we recall the Fundamental Theorem of Calculus (Theorem 7.5.1 on p. 234 of the text), which states that any continuous function f on an interval $[a, b]$ is the derivative of its integral $F(x) = \int_a^x f(t)\,dt$. We will assume this result for now. In particular the function $f(x) = \int_1^x \frac{1}{t}\,dt$ is differentiable with $f'(x) = \frac{1}{x}$ for $x > 0$. It will come as no surprise to you to learn that $f(x)$ is the natural logarithm of x, denoted $\ln x$.

Since the derivative of $\ln x$ is always positive, this function is strictly increasing (as we will see shortly), so that it is one-to-one and has a differentiable inverse. This inverse is none other than your old friend $g(x) = e^x$; the Inverse Function Theorem implies that $g'(x) = g(x)$. Thus without having to assume anything about the exponential function we have shown that there is a strictly increasing function which equals its own derivative. I will later give a different and independent construction of $g(x)$, That functions with positive derivatives are strictly increasing follows at once from the Mean Value Theorem. To prove that theorem we need a result of independent interest, namely the well-known connection between derivatives and maxima or minima of functions.

Theorem 5.2.6, p. 151

Let the differentiable function f on the interval (a, b) have a local maximum or minimum at $x_0 \in (a, b)$. Then $f'(x_0) = 0$.

Proof.

For definiteness assume that x_0 is a maximum; the case where it is a minimum is similar. Taking limits as $x \to x_0^-$, we see that $f'(x_0) \geq 0$; taking limits as $x \to x_0^+$ we see that $f'(x_0) \leq 0$. Hence $f'(x_0) = 0$. □

The same proof shows that if f is differentiable on the *closed* interval $[a, b]$ and has a local maximum at the left-hand endpoint $x = a$, then $f'(a) \leq 0$; likewise if f has a local maximum at the right-hand endpoint $x = b$, then $f'(b) \geq 0$. If instead f has a local minimum at $x = a$ or $x = b$ then these inequalities are reversed.

Rolle's Theorem (5.3.1, p. 156)

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then we have $f'(x) = 0$ for some $x \in (a, b)$.

Proof.

Any such f must have a maximum and a minimum on $[a, b]$; if these both occur at endpoints, then f is constant and there is nothing to prove. Otherwise this result follows at once from the preceding one. □

Now we can prove the Mean Value Theorem on p. 156 (Theorem 5.3.2).

Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is $x_0 \in (a, b)$ with $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Proof.

This follows from Rolle's Theorem applied to the function $g(x) = f(x) - \frac{x-a}{b-a}(f(b) - f(a))$, since $g(a) = g(b) = f(a)$. □

It easily follows that

Corollary 5.3.3, p. 157

A function f is constant on an interval (a, b) if and only if its derivative f' exists and equals 0 there. If f is differentiable on (a, b) with $f'(x) > 0$ there, then f is strictly increasing on (a, b).

Another consequence, this time that you probably have not seen before, is that any derivative satisfies the Intermediate Value Property, even if it is not continuous:

Darboux's Theorem (5.2.7, p. 152)

Let f be differentiable on $[a, b]$. Then f' takes on every value c between $f'(a)$ and $f'(b)$.

Proof.

For definiteness assume that $f'(a) < c < f'(b)$; as usual the other case is similar. Set $g(x) = f(x) - cx$. Then $g'(a) < 0, g'(b) > 0$, whence by a previous remark any minimum of g can only occur at a point $x_0 \in (a, b)$. But g must have a minimum on $[a, b]$ and the result follows. □

If the derivative f' of a differentiable function were always continuous (as you might intuitively expect) then the last result would follow from the Intermediate Value Theorem; but we have already seen that this is not the case. The function f defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$ was previously shown to have derivative equal to $2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$ while $f'(0) = 0$ (see p. 146). What follows from the Intermediate Value Property of derivatives is that derivatives g' cannot have *jump* discontinuities; i.e., that one cannot have $\lim_{x \to a} g'(x)$ existing but different from $g'(a)$. Here, as mentioned last time, $\lim_{x \to 0} f'(x)$ does not exist.