Lecture 5-30: Riemann integration, continued

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Last time I showed that any continuous function on an interval $[a, b]$ is (Riemann) integrable on that interval; now I want to prove the Fundamental Theorem of Calculus, which shows that the integral of any such function is differentiable and in fact an antiderivative of the function itself. I begin with a simple lemma.

Theorem 7.4.1, p. 228

A function f is integrable on an interval $[a, b]$ if and only if it is integrable on $[a, c]$ and $[c, b]$ for any $c \in [a, b]$; in this case we have $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Proof.

Indeed, if f is integrable on $[a, b]$ and $c \in [a, b]$ then for every $\epsilon > 0$ there is a partition P of $[a, b]$ with $U(f, P) - L(f, P) < \epsilon$. Add the point c to P (if necessary) to construct a new partition P'; then $U(f, P') - L(f, P') < \epsilon$, since $L(f, P') \geq L(f, P), U(f, P') \leq U(f, P)$. Intersecting P' with the intervals $[a, c], [c, b]$ yields two partitions P_1, P_2 of $[a, c], [c, b]$, respectively, with $U(f, P_i) - L(f, P_i) < \epsilon$, whence f is integrable on both intervals. Conversely, if f is integrable on both $[a, c]$ and $[c, b]$ and partitions P_1, P_2 of $[a, c], [c, b]$ are chosen so that $U(f, P_i) - L(f, P_i) < \epsilon/2$, then the union P of P_1, P_2 is a partition of $[a, b]$ with $U(f, P) - L(f, P) < \epsilon, U(f, P) = U(f, P_1) + U(f, P_2), L(f, P) = L(f, P_1) + L(f, P_2)$. As ϵ is arbitrary the result follows. □

If $a > b$ and f is integrable on $[b, a]$ then we set $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Then the above formula holds for all a, b, c whenever all relevant integrals are defined.

Now I can prove a strengthened form of one of the Fundamental Theorems of Calculus.

Theorem 7.5.1 (ii), p. 234

Let f be integrable on $[a, b]$ and continuous at $c \in [a, b]$. Set $F(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$. Then F is differentiable at c and $F'(c) = f(c)$.

Proof.

We already know that f is integrable on $[a, x]$, so the definition of $F(x)$ makes sense, and $\frac{F(x)-F(c)}{x-c} = \frac{\int_c^x f(t)\,dt}{x-c}$. This last fraction is bounded between m_x and M_x, where m_x is the infimum and M_x the supremum of f in the interval between c and x. Continuity at c forces $m_x, M_x \to f(c)$ as $x \to c$ and the result follows. □

In particular, if f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for all $x \in [a, b]$, so that f has an antiderivative; this is what Theorem 6.29 actually says. In the special case $f(t) = 1/t$, this completes fully justifies my earlier construction of the natural logarithm $\ln x$: we now know that $1/x$ has an antiderivative, whence I can describe $\ln x$ as the unique antiderivative of $1/x$ taking the value 0 at $x = 1$, As a simple corollary we get

Theorem 7.5.1 (i)

Let f be continuous on $[a, b]$ and F be any antiderivative of f. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof.

The Mean Value Theorem shows that any two antiderivatives of f differ by a constant (so that the conclusion of the theorem is independent of the choice of F). Choosing a particular antiderivative F and setting $G(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$ we have $G(x) = F(x) + c$ for some constant c; plugging in $x = a$, we get the desired result. □

It is also very useful to know that integration is linear.

Linearity of the integral: Theorem 7.4.2 (i),(ii), p. 230

If f, g are integrable on $[a, b]$ and c is constant, then $f \pm g$ and cf are also both integrable, with $\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$, $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

Proof.

The second assertion is clear, since the upper sums of cf are just the multiples by c of the upper or lower sums of f (according as c is positive or negative). Now it suffices to show that $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. For this we just observe that the infimum $m_{c,d}$ of $f+g$ on any interval $[c,d]$ is at least the sum $m'_{c,d} + m''_{c,d}$ of the infima $m'_{c,d}, m''_{c,d}$ of f, g on $[c,d]$; similarly the supremum $M_{c,d}$ of $f+g$ on $[c,d]$ is at most the sum of the suprema $M'_{c,d}, M''_{c,d}$ of f, g on $[c,d]$, whence $L(f+g,P) \geq L(f,P) + L(g,P)$, $U(f+g,P) \leq U(f,P) + U(g,P)$ for any partition P. The result follows. □

I conclude with

Mean Value Theorem for integrals

Let f, g be continuous on $[a, b]$ with $g(x) \geq 0$ for all $x \in [a, b]$. Then there is $c \in [a, b]$ with $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

Proof.

Letting m, M be the minimum and maximum of f on $[a, b]$ we have $mg(x) \leq f(x)g(x) \leq Mg(x)$ on $[a, b]$, whence $\int_a^b f(x)g(x)\,dx$ lies between $m\int_a^b g(x)\,dx$ and $M\int_a^b g(x)\,dx$. By the Intermediate Value Theorem there is $c \in [a, b]$ such that the conclusion holds. □

In particular, taking g to be the constant function 1, we see that for any integrable f we have $\int_a^b f(x)\,dx = f(c)(b-a)$ for some $c \in [a, b]$. Some authors call just this result the Mean Value Theorem for Integrals.