Lecture 5-28: Riemann integration

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For the remainder of the course I will concentrate on (Riemann) integration, as described in Chapter 7 of the text. The basic idea is first to write down a limit defining the area between the graph of a bounded function f on a closed bounded interval $[a, b]$ and the interval $[a, b]$ on the x-axis itself, counting this area as positive whenever the graph of f lies above the x-axis and as negative whenever this graph lies below the axis, then (eventually) to evaluate that limit by antidifferentiating the function f, thereby deducing in particular that every continuous function on $[a, b]$ has an antiderivative (so is the derivative of another function).

To make things as general as possible, we begin with a partition of $[a, b]$, that is, a finite set $P = \{x_0, \dots, x_n\}$ of points $x_0 = a < x_1 < \dots < x_n = b$. Then P divides $[a, b]$ into subintervals $[x_0 = a, x_1], \dots, [x_{n-1}, x_n = b]$. We do not assume that the subintervals $[x_i, x_{i+1}]$ are of equal length. The gap of P is defined to be the largest difference $x_i - x_{i-1}$. For each index i with $1 \le i \le n$ we let m_i, M_i respectively denote the infimum and supremum of f on the interval $[x_{i-1}, x_i]$. It is then intuitively clear that the sums $\sum_{i=1}^n m_i(x_i - x_{i-1}), \sum_{i=1}^n M_i(x_i - x_{i-1})$ are respectively less and greater than the area we are trying to define and evaluate. We denote these sums by $L(f, P), U(f, P)$, respectively, and call them the lower and upper sums of f with respect to P. See Definition 7.2.1 on p. 218.

I will show shortly that $L(f,P) \leq U(f,Q)$ for any partitions P, Q of $[a,b]$; this is Lemma 7.2.4 on p. 219. It then follows that the supremum of the set of all lower sums $L(f,P)$ and the infimum of all upper sums $U(f,P)$ (as P runs through all partitions of $[a,b]$) both exist. We denote these by $\underline{\int}_a^b f(x)\,dx, \overline{\int}_a^b f(x)\,dx$, respectively, and call them the lower and upper integrals of f (see the definition on p. 220 of the text). We say that f is ((Riemann-)integrable if its lower and upper integrals coincide; in this case we denote their common value by $\int_a^b f(x)\,dx$. If $f \leq g$ on $[a,b]$ then it is clear from the definition that $\underline{\int}_a^b f(x)\,dx \leq \underline{\int}_a^b g(x)\,dx$ and similarly with $\underline{\int}$ replaced by $\overline{\int}$; in particular, if both f and g are integrable, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

To show that $L(f, P) \leq U(f, Q)$ for any partitions P, Q of $[a, b]$, note first that it is clear from the definition that $L(f, P) \leq U(f, P)$ for all P. Given $P = \{x_0, \ldots, x_n\}$, let P' be obtained from P by adding one new point y, say between x_{i-1} and x_i. Comparing $L(f, P)$ to $L(f, P')$ we find that the single term $m_i(x_i - x_{i-1})$ is replaced by the sum $m_i'(y - x_{i-1}) + m_i''(x_i - y)$, where m_i', m_i'' are the respective infima of f on $[x_{i-1}, y]$ and $[y, x_i]$; since $m_i \leq m_i', m_i''$ we get $L(f, P') \geq L(f, P)$; similarly $U(f, P') \leq U(f, P)$. By induction we deduce that $L(f, P) \leq L(f, R)$ whenever the partition R of $[a, b]$ contains P as a set; similarly $U(f, P) \geq U(f, R)$ in this situation (Lemma 7.2.4, p. 219). But now since the union $P \cup Q$ of P, Q is a partition of $[a, b]$ whenever P, Q are, we deduce that $L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$, as desired.

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Also f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition P with $U(f, P) - L(f, P) < \epsilon$, since if f is integrable and ϵ is given, we can find partitions P, Q with $U(f, P) - L(f, Q) < \epsilon$; setting $R = P \cup Q$, we get $U(f, R) - L(f, R) \leq U(f, P) - L(f, Q) < \epsilon$, as claimed. Equivalently, f is integrable if and only if there is a sequence P_n of partitions of $[a, b]$ with $\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0$. This is Theorem 7.2.8 on p. 221.

Now consider two extreme examples.

Example

If $f = c$ is a constant function, then it is easy to see that $L(f, P) = c(b - a) = U(f, P)$ for all partitions P of $[a, b]$, whence f is integrable on $[a, b]$ and $\int_a^b f(x)\, dx = c(b - a)$. On the other hand, if $f(x) = 0$ for $x \in \mathbb{Q}$, $f(x) = 1$ for $x \notin \mathbb{Q}$, then for any interval $[a, b]$ we have $L(f, P) = 0, U(f, P) = b - a$ for all partitions P, since every subinterval $[x_{i-1}, x_i]$ contains at least one rational and at least one irrational number. Thus $\underline{\int}_a^b f(x)\, dx = 0, \overline{\int}_a^b f(x)\, dx = b - a$, so that f is not integrable on $[a, b]$. See Example 7.3.3 on p. 225.

Many functions are integrable. In particular we have

Theorem 7.2.9, p. 222

Any continuous function f on an interval $[a, b]$ is integrable.

Proof.

Given $\epsilon > 0$, use the uniform continuity of f (Theorem 4.4.7, p. 133) to choose $\delta > 0$ so that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ whenever $x, y \in [a, b], |x - y| < \delta$. Then given any partition $P = \{x_0, \ldots, x_n\}$ whose gap is less than δ (for example the regular partition $P_n = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, b\}$ for sufficiently large n) we have $U(f, P) - L(f, P) \leq \frac{\epsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon$, whence f is integrable by a previous result. $\square$

Let f be a function on $[a, b]$ and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. For $0 \leq i \leq n-1$, choose a point $y_i \in [x_i, x_{i+1}]$. The weighted sum $R_{f,P} = \sum_{i=0}^{n-1}(x_{i+1} - x_i)f(y_i)$ (which of course depends on the choice of y_i as well as f and P, but this is suppressed from the notation) is called a Riemann sum of f (see Exer. 7.2.6, p. 223). Clearly it lies between $L(f, P)$ and $U(f, P)$. If f is continuous on $[a, b]$, then you have seen that uniform continuity of f implies that, given $\epsilon > 0$ there is $\delta > 0$ such that $U(f, P) - L(f, P)$ for any partition P whose gap is less than δ; then any Riemann sum $R_{f,P}$ corresponding to f and P lies within ϵ of $\int_a^b f(x)\,dx$. With a little more work, it can be shown that the same holds even if f is merely assumed integrable on $[a, b]$, so that if f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx$ is the limit of any sequence of Riemann sums $R(f, P_n)$ corresponding to a sequence (P_n) of partitions of $[a, b]$ such that the gap of P_n goes to 0 as $n \to \infty$.

Note that integrability of f in the last highlighted result is crucial here: the non-integrable function f in the last example is 0 at every rational number, so any Riemann sum $R(P_n, f)$ with a regular partition P_n of the unit interval $[0, 1]$ with the midpoint of each interval $[x_{i-1}, x_i]$ chosen as the point y_i equals 0; but the integral $\int_0^1 f(x)\,dx$ is not defined.

As an example with a continuous f, consider $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n}{n+i+\frac{1}{3}}$. To evaluate this limit, note first that $\frac{n}{n+i+\frac{1}{3}} = \frac{1}{1+\frac{i+\frac{1}{3}}{n}}$. Hence the given sum is a Riemann sum for $f(x) = \frac{1}{1+x}$ on the unit interval $[0, 1]$ corresponding to the regular partition $\{0, 1/n, \ldots, 1\}$ and the point $y_i = \frac{i+\frac{1}{3}}{n}$ in the ith subinterval of this partition. The desired limit is thus $\int_0^1 \frac{1}{1+x}\,dx = \ln 2$.

I conclude by revisiting the Thomae function $h(x)$ that you have seen before: recall that $h(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ in lowest terms and $q > 0$, while $h(x) = 0$ if $x \notin \mathbb{Q}$. We have seen that this function is continuous at all irrational $x \in [0, 1]$ but discontinuous at all rational such x. I claim that $\int_0^1 h(x)\, dx = 0$. To prove this, note first that any lower sum $L(h, P) = 0$, so it suffices to show that the upper integral of h over $[0, 1]$ is 0.

Given $\epsilon > 0$ there are only finitely many positive integers q with $1/q > \epsilon$ and thus only finitely many $y \in [0,1]$ with $h(y) > \epsilon/2$, say $y_1, \ldots, y_n$. Choose a partition $P = \{x_0, \ldots, x_{2n}, x_{2n+1}\}$ of $[0,1]$ such that $y_i \in [x_{2i-1}, x_{2i}]$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n}(x_{2i} - x_{2i-1}) < \epsilon/2$. We have $0 \leq h(x) \leq 1$ for $x \in [0,1]$, whence the total contribution of the terms $M_i(x_{2i} - x_{2i-1})$ to $U(h,P)$ is less than $\epsilon/2$, as is the contribution of the remaining terms to $U(h,P)$, since $h(x) \in [0, \epsilon/2]$ if $x \notin [x_{2i-1}, x_{2i}]$ for any i. Thus $U(h,P) < \epsilon$; since ϵ is arbitrary, we conclude that $\int_0^1 h(x)\,dx = \overline{\int}_0^1 h(x)\,dx = 0$, as claimed.