Lecture 5-21: Review

May 21, 2025

This lecture will be devoted to review for the midterm on Friday. You should review the convergence tests for series, specifically the *n*th term test, which says that any series $\sum a_n$ with $a_n \not\to 0$ as $n \to \infty$ diverges; the comparison test, which says that if $0 \le a_n \le b_n$ for all but finitely many n and $\sum b_n$ converges, then so does $\sum a_n$, while if instead $\sum a_n$ diverges, then so does $\sum b_n$. The limit comparison test vastly enlarges the scope of the straight comparison test; it states that if $a_n, b_n \ge 0$ for all but finitely many n and if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ exists, and if $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge together. Also, the Ratio Test says that if $a_n \ge 0$ for all but finitely many n and $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, then $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$. For any series $\sum a_n$, if $\sum |a_n|$ converges (so that $\sum a_n$ converges *absolutely*, by definition), then $\sum a_n$ converges. If $\sum a_n$ converges but not absolutely, it is said to converge conditionally.

There is essentially only one test for conditional convergence, namely the Dirichlet test, which asserts that if the partial sums of $\sum x_k$ are bounded while $y_1 \geq y_2 \geq \dots$ and $y_k \to 0$ as $k \to \infty$, then $\sum x_k y_k$ converges. In particular, the Alternating Series Test is a special case: if $y_1 \geq y_2 \geq \dots$ and $y_k \to 0$ as $k \to \infty$, then $\sum (-1)^k y_k$ converges.

Next I moved on to functions from $\mathbb{R}$ to itself, the basic limit definition stating that $\lim_{x\to a} f(x) = L$ if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. The value of $f(a)$ itself, even if undefined, is entirely irrelevant to this definition. If $\lim_{x\to a} f(x)$ exists and equals $f(a)$, then we say that f is continuous at a; f is continuous if it is continuous at all points of its domain. Thus for example $g(x) = 1/x$ is continuous even though it is not defined at $x = 0$. With respect to the concepts of open, closed, connected, and compact sets developed in Chapter 3 of the text, f is continuous if and only if the *inverse image* $f^{-1}(U)$ is open for every open subset U of $\mathbb{R}$. If f is continuous and $C \subset \mathbb{R}$ is connected, then so is $f(C)$; likewise if C is compact (closed and bounded), then so is $f(C)$.

In particular, one has two basic properties of continuous functions f on closed bounded intervals $[a, b]$: the Intermediate Value Property, which asserts that for every c between $f(a)$ and $f(b)$ there is $x \in [a, b]$ with $f(x) = c$; and the Extreme Value Property, which asserts that any continuous f on $[a, b]$ has a maximum and minimum value on this interval. Both results are consequences of behavior of continuous functions on connected and compact sets described on the previous slide. Putting them together one sees that $f[a, b]$ is necessarily equal to $[c, d]$ for some closed bounded interval $[c, d]$.

A function f is differentiable at a if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists, in which case this limit is denoted $f'(a)$ and called the derivative of f at a, If f is defined only to the right of a, or only to the left of it, then one extends this definition by replacing the limit above by a suitable one-sided limit. If the derivative f' of f exists everywhere, then f' need not be continuous, but it still satisfies the Intermediate Value Property on any interval over which it is defined. It need *not* however have a maximum or minimum on such an interval, or even be bounded.

The single most important theorem about derivatives is the Mean Value Theorem, which says that if f is continuous on $[a, b]$ and differentiable on (a, b), then for some $c \in (a, b)$ we have $f'(c) = \frac{f(b)-f(a)}{b-a}$. In particular, such an f is weakly increasing on $[a, b]$ if and only if $f'(x) \geq 0$ there, and similarly for weakly decreasing functions. At a local maximum or minimum point x of f on (a, b), we must have either that $f'(x) = 0$ or $f'(x)$ does not exist.

Putting sequences, series, and functions together, I next considered sequences f_n of functions $f_n(x)$ on a subset S of $\mathbb{R}$. One says that the sequence f_n converges pointwise to the function f on S if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$, so that given $x \in S$ and $\epsilon > 0$ there is an index N such that $|f_n(x) - f(x)| < \epsilon$ for any $n > N$; here N is allowed to depend on x as well as ϵ. If more strongly there is a single index N that works simultaneously for all $x \in S$, given ϵ, so that $|f_n(x) - f(x)| < \epsilon$ for all $n > N$ and $x \in S$, then one says that f_n converges uniformly to f on S and calls f the uniform limit of the f_n on S. It is quite possible for the same sequence f_n of functions to converge to its (pointwise) limit uniformly on one subset but not another.

The uniform limit of a sequence of continuous functions on S is again continuous there; this is not necessarily true of a pointwise limit. Even the uniform limit of a sequence of differentiable functions need *not* be differentiable at any point; indeed, any continuous function on $[a, b]$ is the uniform limit of polynomial functions (Weierstrass Approximation,Theorem) but need not be differentiable at any point.

Similarly, one defines the (pointwise) sum f of an infinite series $\sum_{i=1}^{\infty} f_n(x)$ of functions $f_n(x)$ on a set S and says that $\sum f_n(x)$ converges to f uniformly on S if the sequence $s_n(x) = \sum_{i=1}^{n} f_i(x)$ of partial sums of the series converges uniformly to f there. The main technique for producing uniformly convergent series $\sum f_n(x)$ of continuous functions $f_n(x)$ on a set S is the Weierstrass M-test, which states that $\sum f_n(x)$ converges uniformly on S if there is a sequence $M_1, M_2, \ldots$ of constants M_i such that $|f_i(x)| < M_i$ for all $x \in S$ and $\sum M_i$ converges. Then the sum $f(x)$ is continuous on S if the f_i are.

The most important examples of convergent series of functions are Taylor series; that is, series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$. Any such series has a radius of convergence $R \geq 0$ with the property that the series converges absolutely and uniformly if $|x-a| < R$ but diverges if $|x-a| > R$. Whenever the limit $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists, its value equals this radius R. The sum $f(x)$ of a Taylor series $\sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $R > 0$ is infinitely differentiable on the interval $(a-R, a+R)$ and we must have $a_n = f^{(n)}(a)/n!$. Any convergent Taylor series can be differentiated or integrated term by term within the radius of convergence.

Specific Taylor series that you should know for the midterm are the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, valid for $|x| < 1$; the integrated series $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, valid for $-1 \leq x < 1$; the exponential series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, valid for all x; and the trigonometric series
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, again valid for all x. Other convergent series can be obtained from these by changing variables, replacing $x - a$ by $c(x-a)^k$ throughout, where c is a constant and k is a positive integer.

The logisitics of the tes are the same as last time; you are permitted one sheet (front and back) of handwritten notes.