Lecture 5-2: Derivatives

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Just one more property of continuous functions to discuss before we get to derivatives, the topic of Chapter 5 in the text.

Definition 4.4.4, p. 132

We say that the function f defined on a set S is *uniformly continuous on* S if for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $x, y \in S$ and $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

The key difference between this property and ordinary continuity is that, given $\epsilon > 0$, the *same* δ has to work for all $x, y \in S$; we are not allowed to use different δ for different x. (The definitions of continuity and uniform continuity, when written out in logical form, differ only in the order of two quantifiers). For example, the function $f(x) = 1/x$ is continuous but not uniformly continuous on the open interval $(0, 1)$: taking $\epsilon = 1$, we have $f(1/n) = n, f(1/2n) = 2n$ for all $n \in \mathbb{N}$. If any $\delta > 0$ satisfied the above definition for this value of ϵ, then we could choose n with $(1/2n) < \delta$, and then the values $x = 1/n, y = 1/2n$ would contradict this definition.

It turns out that restricting to functions defined on closed bounded intervals fixes the problem.

Theorem 4.4.7, p. 133

Any function f defined and continuous on a closed bounded interval $[a, b]$ is uniformly continuous there.

Proof.

Given $\epsilon > 0$, I will show that the choice $\delta = 1/n$ must satisfy the definition for some choice of $n \in \mathbb{N}$. Otherwise for each such n I would have $x, n, y_n \in [a, b]$ with $|x_n - y_n| < 1/n, |f(x_n) - f(y_n)| > \epsilon$. Then some subsequence x_{n_k} of x_n would converge to some $x \in [a, b]$ and the condition $|x_n - y_n| < 1/n$ forces the corresponding subsequence y_{n_k} of y_n to converge to the same number x. But then $f(x_{n_k}, f(y_{n_k})$ converge to the same limit $f(x)$, contradicting $|f(x_n) - f(y_n)| > \epsilon$ for all n. □

I won't actually have occasion to use uniform continuity in the course until later, but the very similar notion of uniform convergence will play a major role shortly. Uniform continuity will also play a role in the theory of integration.

Now we turn to the basic definition of Chapter 5.

Definition 5.2.1, p. 148

Given a function f defined on an open interval (a, b) and $x_0 \in (a, b)$ we say that f is *differentiable at* x_0 if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. The value of this limit is denoted $f'(x_0)$.

If f is differentiable at x_0 then it is also continuous there (Theorem 5.2.3, p. 148), since then $\lim_{x \to x_0}(f(x) - f(x_0)) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x - x_0) = f'(x_0) \cdot 0 = 0$. A simple application of the difference of powers formula $x^n - x_0^n = (x - x_0)(x^{n-1} + x^{n-2}x_0 + \ldots + x_0^{n-1})$ on p. 90 shows that the power function $f(x) = x^n$ is differentiable everywhere and has derivative $f'(x) = nx^{n-1}$, if $n \in \mathbb{N}$. In fact a slightly more elaborate argument shows for any $r \in \mathbb{R}$ that if $f(x) = x^r$ for $x > 0$ then $f'(x) = rx^{r-1}$ for $x > 0$; here x^r is defined as the supremum of all powers $x^{m/n}$ as m/n runs through the rational numbers less than r, if $x > 1$, and via reciprocals if $0 < x < 1$.

On the other hand, the function $g(x) = |x|$ fails to be differentiable at $x = 0$, since the left-hand limit $\lim_{x \to 0^-} \frac{|x|}{x} = -1$ fails to coincide with the right-hand limit $\lim_{x \to 0^+} \frac{|x|}{x} = 1$. That is, as x runs through real numbers less than 0 but arbitrarily close to it, the difference quotient approaches -1, while as x runs through real numbers greater than 0 but arbitrarily close to it, the difference quotient approaches 1. See Definition 4.6.2 on p. 141.

The limit laws for addition and subtraction show at once that the function $f+g$ is differentiable at any point x_0 if f, g are differentiable there and $(f+g)'(x_0) = f'(x_0) + g'(x_0), (f-g)'(x_0) = f'(x_0) - g'(x_0)$. The calculation $\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x)}{x-x_0} + \frac{f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$ shows that fg is differentiable at x_0 whenever f and g are and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. Similarly we have the quotient rule that $\frac{n}{d}$ is differentiable at x_0 whenever n, d are and $d(x_0) \neq 0$; then $(\frac{n}{d})'(x_0) = \frac{d(x_0)n'(x_0)-n(x_0)d'(x_0)}{d(x_0)^2}$.

The Chain Rule, stated as Theorem 5.2.5 on p. 150, is a bit trickier. Let f be differentiable at $g(x_0)$ and g be differentiable at x_0 (so that f is defined in some open interval containing $g(x_0)$ while g is defined in some open interval containing x_0). Then the composite function $f(g)$ is differentiable at x_0 and $f(g)'(x_0) = f'(g(x_0))g'(x_0)$. To prove this we must study the limit $\lim_{x\to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0}$. There are two cases. If $g'(x_0) \neq 0$ then we must have $g(x) \neq g(x_0)$ for all x in some interval $(x_0 - a, x_0 + a)$ about x_0. For such x we may then write $\frac{f(g(x))-f(g(x_0))}{x-x_0} = \frac{f(g(x)-f(g(x_0))}{g(x)-g(x_0)} \frac{g(x)-g(x_0)}{x-x_0}$ for $x \in (x_0 - a, x_0 + a)$ and the result follows at once by taking limits.

If $g'(x_0) = 0$ then given $\epsilon > 0$ there is $\delta > 0$ such that we still have $|\frac{f(g(x))-f(g(x_0))}{x-x_0} - f'(g(x_0)g'(x_0)| = |\frac{f(g(x))-f(g(x_0))}{x-x_0}| < \epsilon$ whenever $|x - x_0| < \delta$ and $g(x) \neq g(x_0)$, using the continuity of f at $g(x_0)$ and g at x_0; but if $g(x) = g(x_0)$, then trivially $\frac{f(g(x))-f(g(x_0))}{x-x_0} = 0$ for $x \neq x_0$. Thus $f(g)'(x_0) = 0 = f'(g(x_0))g'(x_0)$ in this case too.

Example

Set $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$ (see p. 146). Combining the product and chain rules, we get that $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$, while a direct calculation using the definition of limit shows that $f'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = 0$, since $\frac{x^2}{x} = x$ has the limit 0 as $x \to 0$, while $\sin(1/x)$ is bounded between 1 and -1 for all x (we are applying the squeeze limit law here). This example is interesting because f' always exists but is discontinuous at 0; since $\cos(1/x)$ has no limit as $x \to 0$, the limit $\lim_{x \to 0} f'(x)$ does not even exist. We will return to this example later.