Lecture 5-16: Two counterexamples and one more power series; Weierstrass Approximation Theorem

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By way of contrast with Taylor series I now give an example of a function defined by a uniformly convergent series that is continuous everywhere but differentiable nowhere; this example is essentially the same as the one in the last section of Chapter 5 (pp. 162,3) in the text. Let $f(x) = f_0(x)$ be the distance from x to the nearest integer, so that $f(x) = x$ for $x \in [0, 1/2]$ while $f(x) = 1 - x$ for $x \in [1/2, 1]$; note that $f(x)$ is periodic with period 1. For every positive nonnegative integer i let $f_i(x) = f(4^i x)$, the distance from $4^i x$ to the nearest integer. Finally let $g(x) = \sum_{i=0}^{\infty} 4^{-i} f_i(x)$.

Since $0 \leq f_i(x) \leq 1/2$ for all x, the ith term of this sum is bounded in absolute value by $4^{-i}/2$, whence the series converges (absolutely and) uniformly to a continuous function, by the Weierstrass M-test. I claim that f is not differentiable at any $x \in \mathbb{R}$. To prove this it suffices to produce a sequence x_i converging to x such that the difference quotient $\frac{g(x)-g(x_i)}{x-x_i}$ has no limit as $i \to \infty$. To this end, use decimal notation in base 4 to write any $x \in \mathbb{R}$ as $n + y$ for some $n \in \mathbb{Z}$ and $y = 0.d_1 d_2 \ldots = \sum_{i=1}^{\infty} d_i 4^{-i}$, where each d_i is $0, 1, 2,$ or 3. To avoid ambiguity, do not allow $d_i = 3$ for all but finitely many i, replacing any such expansion by the equivalent one for which $d_i = 0$ for all but finitely many i. Set $y_i = y \pm 4^{-i}$, where the sign is $+$ if $d_i = 0$ or 2, while it is $-$ if $d_i = 1$ or 3; then set $x_i = n + y_i$. Then it is easy to check that $f_j(x_i) = f_j(x) \pm 4^{j-i}$ if $j \leq i$, while $f_j(x_i) = f_j(x)$ if $j > i$.

It follows that the difference quotient $\frac{g(x)-g(x_i)}{x-x_i}$ is a sum of i terms, each ± 1, so is an even integer if i is even and an odd integer if i is odd. But no sequence alternating between even and odd integers can possibly converge (if it did, its limit L would have to be within say $1/3$ of both an even and an odd integer), whence $g'(x)$ exists for no x, as claimed. It is also not difficult to check that g is not monotone on any interval $[a, b]$ with $a < b$. Roughly speaking, the graph of g is infinitely krinkly. The kinks in the graph of each f_i, preventing it from being differentiable at more and more points as $i \to \infty$, combine to destroy the differentiability of g at any point. In fact there are other examples of infinite series $f(x) = \sum_i f_i(x)$ such that each f_i is differentiable everywhere and yet f is still differentiable nowhere.

To continue with my parade of horrors, I now present an example of a function that is infinitely differentiable everywhere but which fails to be analytic at $x = 0$ (that is, does not admit a convergent expansion in powers of x). See the example on p. 203 of the test. Set $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0, f(0) = 0$. The chain rule shows that f is indeed infinitely differentiable at all $x \neq 0$, with its nth derivative $f^{(n)}$ taking the form $p_n(1/x)e^{-\frac{1}{x^2}}$ for some polynomial p_n. What about the point $x = 0$? Taking the limit of $p_n(1/x)e^{-\frac{1}{x^2}} = \frac{p_n(1/x)}{e^{1/x^2}}$ as $x \to 0$ is equivalent to taking the limit of $\frac{p_n(y)}{e^{y^2}}$ as $y \to \infty$; applying L'Hopital's Rule several times, we see that this last limit is 0 for any polynomial p_n. Hence all derivatives of f exist at 0 as well and are equal to 0. The Taylor series of f at $x = 0$ is the 0 series, which clearly does not converge to f. There are worse examples of infinitely differentiable functions g that are not analytic at any point.

There is one more important power series arising often in applications, namely the binomial series, or Newton's binomial expansion. To motivate this series, recall first the binomial theorem, which asserts that $(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m$ for all positive integers n. Note that one formula for the coefficient $\binom{n}{m}$, namely $\frac{n!}{m!(n-m)!}$, makes no sense if the exponent n is replaced by an arbitrary real number α; but an alternative formula, namely $\frac{n(n-1)\ldots(n-m+1)}{m!}$, does make sense with α in place of n.

Accordingly I define the binomial series as $1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n$. The ratio $|\frac{a_n}{a_{n+1}}|$ between the absolute values $|a_n|, |a_{n+1}|$ of the coefficients a_n, a_{n+1} of x^n, x^{n+1} in this series is then $\frac{n+1}{|\alpha-n|}$, which approaches 1 as $n \to \infty$, for any α so this series has radius of convergence 1. It can be shown that this series also converges at $x = \pm 1$, provided that $\alpha > 0$. For example, take $\alpha = 1/2$. The coefficient of x^n in the resulting series is then $\frac{1 \cdot 3 \cdots 2n-3}{2^n n!}(-1)^{n-1} = \frac{1 \cdot 3 \cdots 2n-3}{2 \cdot 4 \cdots 2n}(-1)^{n-1}$ for $n \geq 2$. Plugging in $x = -1$, we get the series $-\sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdots 2n-3}{2 \cdot 4 \cdots 2n}$. Now an easy induction on n shows that $\frac{1 \cdot 3 \cdots 2n-1}{2 \cdot 4 \cdots 2n} < \frac{2}{\sqrt{2n+1}}$ for all $n \geq 1$, whence $\frac{1 \cdot 3 \cdots 2n-3}{2 \cdot 4 \cdots 2n} < \frac{2}{(2n-1)\sqrt{2n+1}}$ and the series for $(1-1)^{1/2}$ converges (necessarily to 0).

To justify the series, I differentiate $f(x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n$ term by term and multiply the differentiated series by $1+x$, which can be regarded as a power series with only two nonzero terms. Recall that I showed earlier that the product $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$ of two absolutely convergent power series $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ is $\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for all n. In the present case, I get $(1+x)f'(x) = \alpha f(x)$, whence $(f(x)(1+x)^{-\alpha})' = 0$ and $f(x) = c(1+x)^{\alpha}$ for some constant c; plugging in $x = 0$, I show that $c = 1$.

Replacing x by $-x^2$ in the series for $(1+x)^\alpha$ and taking $\alpha = -1/2$, I get the series $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1) x^{2n}}{2^n n!}$ for $(1-x^2)^{-1/2}$, valid for $|x| < 1$. Integrating term by term and recalling the Inverse Function Theorem I get the series for $\arcsin x$, the inverse function of $\sin x$ (defined on $[-1,1]$ and taking values in $[0, \pi/2]$), namely $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1) x^{2n+1}}{2^n (2n+1) n!}$. As with the series for $\arctan x$, we pick up $x = \pm 1$ as two more points of convergence of this series. Plugging in these values, I get two series converging absolutely to $\pm\pi/2$, respectively. These series converge much faster than the series $4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ for π derived earlier.

Continuing with the binomial series for $f(x) = (1+x)^{1/2}$ discussed earlier¡ I know by Abel's Theorem and the above calculation that this series converges absolutely and uniformly for $x \in [-1, 1]$. Taking partial sums of this series, it follows that the function $f(x)$ is the uniform limit of a sequence of polynomials on the interval $[-1, 1]$. Setting $x = y^2 - 1$ and observing that a polynomial in x remains polynomial if the variable x is replaced by $y^2 - 1$, I deduce that $g(x) = \sqrt{x^2} = |x|$ is the uniform limit of a sequence of polynomials on this interval. Making a linear change of variable, it follows that for any closed interval $[a, b]$ the function $g(x) = |x|$ is the uniform limit of a sequence of polynomials on this interval.

Making further changes of variable and taking linear combinations, I get that any finite linear combination $h(x) = \sum_{i=0}^{n} b_i|x - a_i|$ for constants a_i, b_i is the uniform limit of a sequence of polynomials on any closed interval $[a, b]$ containing all the a_i. Now any such combination (restricted to $[c, d]$) is a polygonal function in the sense that it is continuous and its graph in the xy-plane is a finite union of line segments. Conversely, I claim that any polygonal function is a linear combination of absolute values as above. To prove this, let f be polygonal and defined on the interval $[c, d]$. Let $a_0 = c < a_1 < \cdots < a_n = d$ be the x-coordinates of the endpoints of the intervals making up the graph of f. Given a combination $g = \sum_{i=0}^{n} b_i|x - a_i|$, one checks for $1 \leq i \leq n - 1$ that $2b_i$ is the difference in slopes of the graph of g between a_{i-1} and a_i and between a_i and a_{i+1}. Choose b_i for $1 \leq i \leq n - 1$ so that $2b_i$ matches the corresponding difference in slopes for the graph of f.

Next, choose b_0 so the value $g(a_n) = \sum_{i=0}^{n-1} b_i(a_n - a_i)$ of g at a_n matches $f(a_n)$; finally, choose b_n so that the slope $b_0 - \sum_{i=1}^{n} b_i$ of the graph of g between a_0 and a_1 matches that of f. The upshot is then that $g(x) = f(x)$ on $[c, d]$, so that any polygonal function is the uniform limit of a sequence of polynomials on its domain $[c, d]$; equivalently, given a polygonal function f on $[c, d]$ and $\epsilon > 0$ there is a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all $x \in [c, d]$.

Finally, let f be any continuous function on an interval $[a, b]$ and let $\epsilon > 0$. Then f is uniformly continuous on $[a, b]$, so there isn integer n such that $|f(x) - f(y)| < \epsilon/4$ whenever $x, y \in [a, b]$ and $|x - y| < \frac{b-a}{n}$. Let g be the unique polygonal function on $[a, b]$ such that the endpoints of the segments in its graph are $(a + i\frac{b-a}{n}, f(a + i\frac{b-a}{n}))$ for $0 \leq i \leq n$. One easily checks that $|f(x) - g(x)| < \epsilon/2$ for $x \in [a, b]$. Then choose a polynomial p such that $g(x) - p(x)| < \epsilon/2$ for $x \in [a, b]$. I finally deduce

Weierstrass Approximation Theorem (6.7.1, p. 206)

Given a continuous function f on $[a, b]$ and $\epsilon > 0$, there is a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. Equivalently, f is the uniform limit on $[a, b]$ of a sequence of polynomials.

In particular, even the monstrous function f above that is continuous everywhere and differentiable nowhere is a uniform limit of (a sequence of) polynomials when restricted to any interval $[a, b]$. Note however that it is too much to expect even a very nice (but nonpolynomial) function to be a uniform limit of polynomials on the unbounded interval $[0, \infty]$; in fact, not even the exponential function e^x is such a limit. If it were, there would in particular be a polynomial p such that $|p(x) - e^x| < 1$ for all $x \geq 0$; but this would imply that $\lim_{x\to\infty} \frac{p(x)}{e^x} = 1$, whereas we have seen that any such limit is 0.