

Lecture 5-14: Power and Taylor series: examples

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Last time you learned that any Taylor series

$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ with positive (or infinite) radius of convergence R can be integrated term by term within the radius of convergence, so that the series $g(x) = \sum_{k=0}^{\infty} \frac{a_k(x-a)^{k+1}}{k+1}$ converges for $|x-a| < R$ and $g(x) = \int_a^x f(t) dt$ for all x in this range. I also showed that the differentiated series $h(x) = \sum_{k=1}^{\infty} k a_k(x-a)^{k-1}$ also has radius of convergence R and its term-by-term integrated series coincides with $f(x)$, so that $f(x)$ is differentiable and $f'(x) = h(x)$ for $|x-a| < R$.

A very handy result due to the Norwegian mathematician Abel asserts that if a Taylor series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converges at $a + R > a$, then the convergence is uniform on the entire closed interval $[a, a + R]$, so that the sum is a continuous function on that interval (Theorem 6.5.4, p. 193). This result follows from the Dirichlet test for convergence of a series, which shows that the partial sums of $\sum_{n=0}^{\infty} a_n(x - a)^n$ are uniformly Cauchy for $x \in [a, a + R]$. Here we are writing the series $\sum_{n=0}^{\infty} a_n(x - a)^n$ as $\sum_{n=0}^{\infty} (a_n R^n) \left(\frac{x-a}{R}\right)^n$ to apply the Dirichlet test. The same result holds if the series converges at $a - R < a$, replacing $[a, a + R]$ by $[a - R, a]$. Thus, for example, since the series for $-\ln(1 - x)$, namely $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ has radius of convergence 1 and also converges (by the Alternating Series Test) at $x = -1$, its sum at that point must be $\lim_{x \rightarrow -1^+} \ln(1 - x) = -\ln 2$.

In a similar way, the geometric series for $\frac{1}{1+x^2}$, namely $\sum_{n=0}^{\infty} (-1)^n x^{2n}$, which has radius of convergence 1, may be integrated term by term to the series $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, which also has radius of convergence 1. This time this series also converges at both $x = \pm 1$, so its sums at those points must be $\pm\pi/4$.

Example

The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$, say to $f(x)$. Since the term-by-term differentiated series is the same series, we have $f'(x) = f(x)$, whence this function f equals its own derivative. We therefore define $f(x) = e^x$ by this series; we can even show by manipulating series that $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$. Next consider the closely related series

$s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Here one checks by the ratio test that both series have infinite radius of convergence; in applying this test to $s(x)$, for example, one should of course look at the ratio of two successive *nonzero* terms

$\frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$, so that one does not divide by 0. By differentiation one gets $s'(x) = c(x)$, $c'(x) = -s(x)$, whence by differentiation one gets that $s(x)^2 + c(x)^2$ is constantly equal to its value at 0, namely 1, since the derivative of this quantity is 0. I define $\sin x = s(x)$, $\cos x = c(x)$. Note that

$s''(x) = -s(x)$, $c''(x) = -c(x)$. In this way I have defined the sine

Another simple but very useful way to get new Taylor series from old ones is to change the variable, replacing $x - a$ by $(x - a)^k$ throughout for some fixed positive integer k . Thus, starting with the series $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, valid for all x , we get the series $\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$, also valid for all x . We can also multiply variables by constants: since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x , we also get $e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ for all x . In the case of series with finite radii of convergence, however, we must be careful to stay within the interval of convergence. Thus the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ for $|x| < 1$; replacing x by $3x$, we get the series $\sum_{n=0}^{\infty} 3^n x^n$ for $\frac{1}{1-3x}$; but this series converges only for $|x| < 1/3$.

Example

I can also work out power series expansions for functions that I cannot even explicitly write down. For example, starting from the geometric series $\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}$, valid for all $x \in (-1, 1)$, I get the series $\int_0^x \frac{1}{1+t^3} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}$, valid for all x in the larger interval $[-1, 1]$, even though there is no formula for this integral.

Adding the series for $\ln(1+x)$ and $-\ln(1-x)$, I get the series $2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$, which converges to $\ln \frac{1+x}{1-x}$ for $|x| < 1$. As it turns out that any positive number y can be written as $\frac{1+x}{1-x}$ for some $x \in \mathbb{R}$ with $|x| < 1$, I now have a convergent series which can be used to compute the natural logarithm of any positive number.

In general, given any function $f(x)$ which admits a Taylor series expansion $\sum_{n=0}^{\infty} a_n(x-a)^n$ with a positive radius of convergence, I can differentiate this series n times and plug in $x = a$ to deduce that $a_n = f^{(n)}(a)/n!$ for all n , where $f^{(n)}(a)$ denotes the n th derivative of f at a . This formula leads to a natural definition.

Definition; cf. Theorem 6.6.2, p. 199

Given an infinitely differentiable function $f(x)$ on an interval $(a-R, a+R)$ for some $R > 0$, its *Taylor series at $x = a$* is the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$. This is the only Taylor series at $x = a$ that has a chance of converging to $f(x)$ for $x-a \in (-R, R)$. If it does converge to $f(x)$ on the interval $(a-R, a+R)$ (for some $R > 0$) we say that f is *analytic at $x = a$* .

Unfortunately it is in the nature of things that not all infinitely differentiable functions are analytic at all points. For example, it is known that **the Taylor series at $x = a$ of an infinitely differentiable function can be an arbitrary Taylor series**, so that there is no guarantee that this series will converge at any point other than $x = a$. Secondly, even if the Taylor series of a function f at $x = a$ does converge, it can converge to a function different from f . I will explore these matters in more detail later.