Lecture 4-9: Sequences and series of real numbers

April 9, 2025

I will begin by showing that sequences behave in the way one would expect under arithmetic operations.

Limit law for addition: Theorem 2.3.3 (ii), p. 50

If the sequences s_n, t_n converge to L, M, respectively, then $s_n + t_n$ converges to $L + M$.

Proof.

This proof is called an $\epsilon/2$ proof, for reasons that will shortly become clear. Given $\epsilon > 0$, choose indices N_1, N_2 such that $|s_n - L), |t_n - M|$ are both less than $\epsilon/2$ for $n \geq N_1, N_2$, and let $N = \max\{N_1, N_2\}$. For any $n \geq N$ we then have $|s_n + t_n - L - M| \leq |s_n - L| + |t_n - M| < (\epsilon/2 + \epsilon/2) = \epsilon$ by the Triangle Inequality, showing that $s_n + t_n \to L + M$ as $n \to \infty$, as desired. □

The same argument shows that $s_n - t_n \to L - M$ as $n \to \infty$.

Next we have

Limit law for multiplication: Theorem 2.3.3 (iii)

If $s_n \to L$, $t_n \to M$, as $n \to \infty$, then $s_n t_n \to LM$.

Proof.

This is another $\epsilon/2$ proof but slightly more complicated than the previous one. Begin by observing that $s_n t_n - LM = s_n(t_n - M) + (s_n - L)M$. Given $\epsilon > 0$, choose N_1, N_2 so that $n \geq N_1$ implies $|s_n - L||M| < \frac{\epsilon}{2}$; note that any N_1 works if $M = 0$, while if $M \neq 0$ one can choose such that $n \geq N_1$ implies $|s_n - L| < \frac{\epsilon}{2|M|}$. Next choose an index N_2 such that $n \geq N_2$ implies that $|s_n - L| < 1, |s_n| < |L| + 1$; finally choose N_3 such that $n \geq N_3$ implies $|t_n - M| < \frac{\epsilon}{|L|+1}$. Set $N = \max\{N_1, N_2, N_3\}$. The triangle inequality then shows that $n \geq N$ implies that $|s_n t_n - LM| = |s_n(t_n - M) + (s_n - L)M| \leq |s_n||t_n - M| + |s_n - L||M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, as desired. □

In particular, if $s_n \to L$ as $n \to \infty$ then we have $cs_n \to cL$ as $n \to \infty$ for any $c \in \mathbb{R}$.

Finally, we have

Limit law for reciprocals

If $s_n \to L$ as $n \to \infty$ and $L \neq 0$ then $\frac{1}{s_n} \to \frac{1}{L}$ as $n \to \infty$.

Proof.

First of all, taking $\epsilon = \frac{|L|}{2}$, we have an index N such that $|s_n - L| < \frac{|L|}{2}$ for $n \geq N$, whence in particular $|s_n| > \frac{|L|}{2}$ for any $n \geq N$ and $\frac{1}{s_n}$ is at least defined for $n \geq N$. Next, given any $\epsilon > 0$ there is an index N_1 such that $n \geq N, N_1$ implies $|s_n - L| < \epsilon \frac{L^2}{2}$, whence we also have $|\frac{1}{s_n} - \frac{1}{L}| = \frac{|s_n - L|}{|s_n L|} < \frac{|s_n - L|}{L^2/2} < \epsilon$. Taking $N_2 = \max\{N, N_1\}$ we deduce that $|\frac{1}{s_n} - \frac{1}{L}| < \epsilon$ for $n > N_2$ and so $\frac{1}{s_n} \to \frac{1}{L}$, as desired. □

Combining the last two results we immediately get

Limit law for division: Theorem 2.3.3 (iv)

If $s_n \to L$, $t_n \to M$ with $M \neq 0$, then $\frac{s_n}{t_n} \to \frac{L}{M}$ as $n \to \infty$.

Another very useful result is

Sandwich or squeeze theorem: Exercise 2.3.3, p. 54

If $s_n \leq t_n \leq u_n$ for all n and $s_n \to L$, $u_n \to L$ as $n \to \infty$, then $t_n \to L$ as $n \to \infty$.

Proof.

Indeed, given $\epsilon > 0$ we have indices N_1, N_2 such that $n \geq N_1, N_2$ implies $|s_n - L|, |u_n - L| < \epsilon$, whence both s_n, u_n lie in the open interval $(L - \epsilon, L + \epsilon)$, forcing $t_n \in (L - \epsilon, L + \epsilon), |t_n - L| < \epsilon$ for $n \geq \max\{N_1, N_2\}$, as desired. □

Example

One has $\frac{\sin n}{n} \to 0$ as $n \to \infty$ since $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ and $\frac{-1}{n}, \frac{1}{n} \to 0$ as $n \to \infty$.

Example

In homework you have shown that every positive real number has a (unique) square root. I will show later that every positive number x has a unique positive nth root, for every positive integer n. Denoting this root as usual by $x^{1/n}$ one then has $n^{1/n} \to 1$ as $n \to \infty$. To see this set $x_n = n^{1/n} - 1$. The binomial theorem then shows that $n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$, whence $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$. The squeeze theorem then shows $x_n \to 0$ as $n \to \infty$, as claimed.

A slightly more complicated calculation shows that

Example

For any $p > 0$ and positive integer k one has $\frac{n^k}{(1+p)^n} \to 0$ as $n \to \infty$. Choose an integer $m > k$. For $n > 2m$ one has $(1+p)^n > \binom{n}{m} p^m = \frac{n(n-1)\cdots(n-m+1)}{m!} p^m > \frac{n^m p^m}{2^m m!}$, whence $0 < \frac{n^k}{(1+p)^n} < \frac{2^m m!}{p^m} n^{k-m}$ for $n > 2m$. Since I have showed that $n^{k-m} \to 0$ as $n \to \infty$ (since $k - m < 0$) the result follows.

I will conclude with the geometric series $\sum_{n=0}^{\infty} x^n$, which converges to $\frac{1}{1-x}$ if $|x| < 1$ and diverges otherwise (see p. 73). Indeed, it is clear from previous examples that this series diverges if $x = \pm 1$. For $x \neq 1$ one has $\sum_{i=0}^{n} x^n = \frac{1-x^{n+1}}{1-x}$; this last expression has no finite limit if $|x| > 1$ but it has the limit $\frac{1}{1-x}$ if $|x| < 1$.