Lecture 4-7: Sequences of real numbers

April 7, 2025

Last week I showed that in order to make sense of an infinite sum $\sum_{i=1}^{\infty} a_i$ it is necessary to consider the *partial sums* $s_n = \sum_{i=1}^{n} a_i$, which from a succession $s_1, s_2, \ldots$ of numbers. If this succession "crunches down" on some number s in some sense, then that number s deserves to be called the value of the sum $\sum_{i=1}^{\infty} a_i$.

But now what does it mean for succession of numbers to crunch down on another one? To answer this I first define the notion of succession formally.

Definition 2.2.1, p. 42

A *sequence* (of real numbers) is a function s from the natural numbers to the real numbers. I write s_n rather than $s(n)$ for the value of s at n and denote the entire sequence by $\{s_n\}$, or just s_n.

One often gives a formula for s_n, either directly as a function of n or inductively; for example, there is the sequence of reciprocals given by $s_n = 1/n$ and the famous *Fibonacci sequence* given by $s_1 = s_2 = 1, s_n = s_{n_1} + s_{n-2}$ for $n \geq 3$. Although there is in fact a formula for s_n in terms of n in this last example, it is important to realize that such a formula is not actually necessary to define the sequence (s_n); for this the inductive recipe $s_n = s_{n-1} + s_{n-2}$ is enough.

Definition 2.4.3, p. 57

An *infinite series* is an expression of the form $\sum_{i=1}^{\infty} a_i$, where the a_i are real; sometimes I allow the index of summation to start at a different number than 1. I will identify any such series with its sequence of partial sums s_n, where $s_n = \sum_{i=1}^{n} a_i$.

Definition 2.2.3, p. 43

I say that the sequence $\{s_n\}$ *converges to the limit* L if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ for all $n \geq N$. I write $\lim s_n = \lim_{n\to\infty} s_n = L$ or $s_n \to L$ (and say s_n approaches L) as $n \to \infty$ in this situation. I can also say that $\{s_n\}$ converges without specifying its limit. If instead for every M there is an index N such that $s_n > M$ for $n \geq N$, then I say that s_n *diverges to* ∞ and write $s_n \to \infty$ as $n \to \infty$. I define $s\_n \to -\infty$ as $n \to \infty$ similarly. An infinite series $\sum_{i=1}^{\infty} a_i$ converges (by definition) if and only if the sequence $(s_n = \sum_{i=1}^{n} a_i)$ does. If so, the limit of (s_n) is called the sum of the series $\sum_{i=1}^{\infty} a_i$. If instead $s_n \to \infty$ as $n \to \infty$, then I may write $\sum_{i=1}^{\infty} a_i = \infty$, but I do not use the term “converge” in this situation.

For example, the sequence with $a_n = (-1)^n/n$ converges to 0: given $\epsilon > 0$, choose $N > 1/\epsilon$ by the Archimedean Property; then $|\frac{(-1)^n}{n}| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$ for $n \geq N$, whence $|a_n| < \epsilon$ for $n \geq N$, as desired. Note that 0 is neither the supremum nor the infimum of the set $\{a_n\}$ of terms of the sequence in this case.

On the other hand, the sequence $\{(-1)^n\}$ does not converge (one says it diverges). Indeed, suppose that this sequence converges to a. There would then have to be some N such that $|(-1)^n - a| < \frac{1}{2}$ for all $n \geq N$. If n is odd, this implies that $|-1-a| < \frac{1}{2}$; if n is even, this implies that $|1-a| = |a-1| < \frac{1}{2}$. Applying the Triangle Inequality I get $|-1-1| = 2 < \frac{1}{2} + \frac{1}{2} = 1$, which is absurd. This is the first application of the Triangle Inequality in this course.

Often one can understand a complicated sequence by comparing it to a simpler one. Thus for example one has $\lim_{n\to\infty}(\frac{2}{n^2}+\frac{4}{n}+3)=3$. To see this observe that $\frac{2}{n^2}\leq\frac{2}{n}$ for $n\geq 1$, whence $0\leq\frac{2}{n^2}+\frac{4}{n}\leq\frac{6}{n}$ and $|\frac{2}{n^2}+\frac{4}{n}+3-3|\leq\frac{6}{n}$. Given $\epsilon>0$, if we choose $N>\frac{6}{\epsilon}$ then $\frac{6}{n}<\epsilon$ for $n\geq N$ whence the absolute value is less than ϵ, as desired.

Sometimes clever algebraic manipulations make it easy to evaluate certain limits. Thus for instance if $a_n = \sqrt{n+1} - \sqrt{n}$ then we can rewrite a_n as $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$, whence it easily follows that $a_n \to 0$ as $n \to \infty$. On the other hand, consider the *series* $\sum_{i=1}^{\infty}(\sqrt{i+1} - \sqrt{i})$. This is a so-called telescoping series; its nth partial sum $s_n = \sum_{i=1}^{n}(\sqrt{i+1} - \sqrt{i}) = \sqrt{n+1} - \sqrt{1}$. Clearly $s_n = \sqrt{n+1} - 1 \to \infty$ in the sense that for any M there is N such that $s_n > M$ for $n \geq N$. The same argument shows that even if we begin the series at a different index, say at $i = m$, then the nth partial sum $s_{n,m}$ of this series still approaches ∞ as n does.

This example has the following remarkable consequence. Given $\epsilon > 0$ choose N large enough that $a_n < \epsilon$ for $n \geq N$. The partial sums of $\sum_{i=N}^{\infty} a_i$ approach ∞ as n does, so given any positive real number r there is a smallest partial sum $s_{n,N} = \sqrt{n+1} - \sqrt{N}$ greater than r. But then $s_{n-1,N} = \sqrt{n} - \sqrt{N} < r$, whence $\sqrt{n+1} - \sqrt{N}$ lies between r and $r + \epsilon$. I conclude that the set S of differences $\sqrt{m} - \sqrt{n}$ of square roots of positive integers m, n for $m > n$ is dense in $\mathbb{R}^+$, whence by interchanging m and n we see that the set S' of differences $\sqrt{m} - \sqrt{n}$ of square roots of arbitrary positive integers is dense in $\mathbb{R}$. One might well have guessed that the set S' is discrete; but in fact, like $\mathbb{Q}$, it is dense in $\mathbb{R}$.

The Binomial Theorem implies that $(1+r)^n \geq 1+nr$ for any $r > 0$ and positive integer n, whence $(1+r)^n \to \infty$ as $n \to \infty$; thus $\alpha^n \to \infty$ if $\alpha > 1$. If instead $0 < \alpha < 1$, then $(1/\alpha) > 1$, so $(1/\alpha)^n \to \infty$ and $\alpha^n \to 0$ as $n \to \infty$. I will derive consequences of this calculation for certain infinite series next time.

The Least Upper Bound Property enables me to guarantee that certain sequences converge without specifying their limits. Call a sequence s_n increasing (or monotone increasing) if $s_n \leq s_{n+1}$ for all n; call it bounded if the set $S = \{s_n : n \in \mathbb{N}\}$ of all of its terms is bounded. Then one has

Theorem 2.4.2, p. 56

Any bounded increasing sequence converges.

Indeed, if s_n is bounded and increasing, let x be the supremum of the set S. Given $\epsilon > 0$, the number $x - \epsilon$ is not an upper bound for S, so choose N with $s_N > x - \epsilon$. Then we have $x - \epsilon \leq s_N \leq s_n \leq x$ for all $n \geq N$, so $|s_n - x| < \epsilon$ for all such n and s_n converges to x, as claimed.