Lecture 4-4: The real numbers, concluded

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I will begin by treating decimal expansions more thoroughly than I did on the first day of class. Given one such, say $0.d_1 d_2 d_3 \ldots$, with each d_i an integer between 0 and 9, it denotes the infinite sum $\sum_{i=0}^{\infty} d_i 10^{-i}$. Such an infinite sum is called an infinite series and will be officially discussed only later in the course; but already I have the tools I need to define the sum of this particular series. The intuition is that whatever this sum turns out to be, it should be larger than the truncated sum $\sum_{i=1}^{n} d_i 10^{-i}$ for any n, and in fact the sum should be the least real number larger than all truncated sums. Accordingly I define $\sum_{i=1}^{\infty} d_i 10^{-i}$ to be the least upper bound $\sup S$ of the set $S = \{\sum_{i=1}^{n} d_i 10^{-1} : n \in \mathbb{N}\}$.

Next I have to check that S is bounded above. Indeed, any truncated sum $\sum_{i=1}^{n} d_i 10^{-i} \leq \sum_{i=1}^{n} 9 \cdot 10^{-i} = 1 - 10^{-n}$, by the well-known formula for the sum of a finite geometric series (Example 2.7.5, p. 73). In particular, all truncated sums are less than 1, so S is bounded above by 1 and indeed has a least upper bound.

Conversely, given any real number $x \in [0, 1]$, I can inductively define a decimal expansion $0.d_1 d_2 \ldots$ that equals x. Start by letting d_1 be the largest integer between 0 and 9 with $\frac{d_1}{10} \leq x$; if $d_1, \ldots, d_n$ have been defined, let d_{n+1} be the largest integer between 0 and 9 with $\sum_{i=1}^{n} d_i 10^{-i} + d_{n+1} 10^{-n-1} \leq x$. Then by the construction every sum $x - 10^{-m} < \sum_{i=1}^{m} d_i 10^{-1} \leq x$. Since one easily proves by induction that $10^{-m} < m^{-1}$ for any $m \in \mathbb{N}$ and for any $\epsilon > 0$ we have $m^{-1} < \epsilon$ for some m by the Archimedean Property it follows that x is an upper bound for all the sums $\sum_{i=1}^{m} d_i 10^{-i}$ but $x - \epsilon$ is not, for any $\epsilon > 0$. Hence x is indeed the least upper bound of the set of all such sums, as desired.

The decimal expansion of a real number x between 0 and 1 is unique, apart from the case where x admits one expansion $0.d_1 d_2 \ldots$ ending in a string of 0s (so that there is i with $d_i \neq 0$ but $d_j = 0$ for all $j > i$). In that case there is an equal expansion $0.e_1 e_2 \ldots$ ending in a string of 9s; more precisely, we have $e_j = d_j$ for $j < i$, $e_i = d_i - 1$, and $e_k = 9$ for $k < i$. It is not difficult to check that this is the only situation in which the decimal expansion of a real number is not unique.

I now review the notion of uncountability, showing that the closed interval $I = [0, 1]$ is uncountable, so that there is no surjective map f from $\mathbb{N}$ onto I. (See theorem 1.5.6 on p. 27 and its proof in the text.) If there were such an f, let $x_j = f(j) = \sum_{i=1}^{\infty} d_{ji} 10^{-i}$, the image of $j \in \mathbb{N}$ under f. I need to find a real number $x \in I$ different from x_j for all j. Enlarge the set of x_j if necessary to include all expansions ending in a string of 9s equalling an x_j ending in a string of 0s and vice versa, so that every expansion different from that of any x_j definitely represents a number not equal to any x_j Then for each i choose a digit $e_i \neq d_{ii}$; for example, set $e_i = 0$ if $d_{ii} \neq 0$ and $e_i = 1$ if $d_{ii} = 0$. Then the expansion $x = \sum_{i=1}^{\infty} e_i 10^{-i}$ is different from x_i for all i, as required, so that there is no surjective map f, as claimed.

It turns out that the Least Upper Bound Property of $\mathbb{R}$ is intimately tied up with its uncountability. Indeed, it is *almost*, but not quite, true that if an infinite subset T of $\mathbb{R}$ satisfies the Least Upper Bound Property in the sense that the least upper bound of any bounded subset of T lies in T, then T is uncountable. There is a missing hypothesis: I must also assume that T is dense (not dense in anything else, just dense), in the sense that for any $x, y \in T$ with $x < y$ there is $z \in T$ with $x < z < y$. Indeed the natural numbers $\mathbb{N}$ also satisfy the Least Upper Bound Property, but are countable; this is possible only because $\mathbb{N}$ is *discrete* in the sense that between any two consecutive natural numbers n and $n + 1$ there are no natural numbers.

Now there are many proper subsets T of $\mathbb{R}$ (other than $\mathbb{R}$ itself) satisfying both the Least Upper Bound and Greatest Lower Bound Properties, in the above sense that the supremum of any subset of T that is bounded above lies in T and the infimum of any subset of T bounded below also lies in T; for example, any closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ has this property (but an open or half open interval like $[a, b)$ or (a, b) does not, since it excludes one or both endpoints a, b. A set T with this property is called *closed* in $\mathbb{R}$. It is easy to check that any *finite* union $\cup_{i=1}^{n} T_i$ of closed sets T_i is again closed, as is any intersection $\cap_i T_i$ of closed subsets of $\mathbb{R}$, finite or not. An *infinite* union of closed sets T_i need not be closed. There is also a notion of *open subset* of $\mathbb{R}$, but this is *not* the same as a non-closed subset!

A very important inequality for real numbers is

The Triangle Inequality; Example 1.2.5, p. 8

For $x, y \in \mathbb{R}$ we have $|x+y| \leq |x| + |y|$

Proof.

I have $-|x| \leq x \leq |x|, -|y| \leq y \leq |y|$, whence by addition I get $-(|x|+|y|) \leq x+y \leq |x|+|y|$; this immediately gives the desired result. $\square$

A much deeper fact is that this same result holds in higher dimensions: defining the norm $||\vec{v}||$ of a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to be $\sqrt{x_1^2 + \ldots + x_n^2}$ we have $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$; a proof of this can be found in many books or online. Here one can draw a picture of an actual triangle (with vertices $\vec{0}$, $\vec{x}$, and $\vec{x} + \vec{y}$) to illustrate the result, unlike the situation with $x, y \in \mathbb{R}$.