Lecture 4-30: Extreme and Intermediate Value Theorems

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I will now derive some fundamental properties of continuous functions on closed bounded intervals; these are used constantly in calculus.

Theorem 4.4.1, p. 129

Let $K \subset \mathbb{R}$ be a compact set and f a real-valued function continuous on K. Then the range $f(K)$ of f on K is also compact.

Let y_k be a sequence of points in $f(K)$ with $y_k = f(x_k), x_k \in K$. Then the sequence x_k has a subsequence, say x_{n_k}, converging to $x \in K$, since K is closed, whence $y_{n_k} = f(x_{n_k})$ converges to $y = f(x)$. This simultaneously shows that $f(K)$ must be bounded (since otherwise I could choose the y_k so that no subsequence of it is bounded) and closed, as desired.

In particular, since closed intervals $[a, b]$ are compact, we get that $f[a, b]$ is closed and bounded whenever f is continuous on the interval $[a, b]$. In particular, any such function f takes on a maximum and a minimum on $[a, b]$, since as a closed and bounded set it must contain both its supremum and infimum. Note that this result depends crucially on the closedness of the interval: the functions $f(x) = x$ and $g(x) = 1/x$ are both continuous on the open interval $(0, 1)$, but neither takes on a maximum or minimum value on this interval (and the function g is not even bounded there).

A more striking result with immediate applications is the following.

Theorem 4.5.2, p. 136

Let f be a continuous function on a connected subset C of $\mathbb{R}$. Then the range $f(C)$ is also connected.

Let A, B be disjoint open and closed subsets of $D = f(C)$ whose union is D. Then $f^{-1}(A), f^{-1}(B)$ are disjoint open and closed subsets of C whose union is C, whence one or the other of them is empty. This forces one or the other of A and B to be empty, as claimed.

In particular, since we have seen that the connected subsets of $\mathbb{R}$ are exactly the intervals (open, half-open, or closed, and bounded or unbounded at both ends), it follows that the range of a continuous function on an interval is an interval.

As an immediate consequence of the previous result we get

Intermediate Value Theorem (4.5.1, p. 136)

A continuous function f on a closed bounded interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$; that is, if c lies between $f(a)$ and $f(b)$, then there is $x \in [a, b]$ with $f(x) = c$.

A further remarkable consequence is

Theorem

If f is continuous and one-to-one on an interval $[a, b]$ then it is (strictly) monotone (either strictly increasing or strictly decreasing) on this interval.

Proof.

Otherwise we would have $x, y, z \in [a, b]$ with $x < y < z$ and either $f(x) < f(y) > f(z)$ or $f(x) > f(y) < f(z)$. In either case there is some c simultaneously lying between $f(x)$ and $f(y)$ and between $f(y)$ and $f(z)$ such that f takes the value at least once in the interval (x, y) and again in (y, z). This contradicts f being one-to-one. □

Putting together the Intermediate and Extreme Value Theorems, we see that the range of a continuous function f on a closed bounded interval $[a, b]$ is a closed bounded interval $[c, d]$. If f is one-to-one on $[a, b]$ then we can say more: either $c = f(a), d = f(b)$, in which case f is strictly increasing on $[a, b]$, or $c = f(b), d = f(a)$, in which case f is strictly decreasing on $[a, b]$.

Moreover, we have

Theorem; cf. Exercise 4.5.8, p. 140

If f is continuous and strictly monotonic on $[a, b]$, sending this interval to $[c, d]$, then the inverse function g of f, mapping $[c, d]$ to $[a, b]$, is also continuous.

Indeed, it suffices to show that the inverse image of an open subinterval of $[a, b]$ is an open subinterval of $[c, d]$; but this follows at once from the monotonicity and the Intermediate Value Theorem.

Now we can write down a large family of continuous functions that are inverses of other continuous functions. For example, for every positive integer n, the function f_n sending x to x^n is strictly increasing and continuous on the positive real line $\mathbb{R}^+ = [0, \infty)$. The inequality $(1 + x)^n \geq 1 + nx$, valid for all positive x by the Binomial Theorem, together with the elementary observation $(1/x)^n = 1/x^n$, then shows that the range of this function on $\mathbb{R}^+$ is all of $\mathbb{R}^+$. Hence the inverse function g_n sending x to $x^{1/n}$ is well defined and continuous on $\mathbb{R}^+$, mapping it onto $\mathbb{R}^+$. If n is odd, then f_n is strictly increasing on all of $\mathbb{R}$, whence g_n is well defined, continuous, and increasing on $\mathbb{R}$ as well. For any rational number m/n, we can then define $x^{m/n}$ to be $(x^{1/n})^m$ for any $x \in \mathbb{R}^+$. This function is well defined, increasing, and continuous on $\mathbb{R}^+$.

Another beautiful consequence of the Intermediate Value Theorem is that a large number of equations involving continuous functions are guaranteed to have solutions, even if we cannot write down these solutions explicitly. For example, consider the equation $\cos x = x$. There is no formula for any $x \in \mathbb{R}$ satisfying this equation; but if we set $f(x) = \cos x - x$, then f is a continuous function with $f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 < 0$; so there must be some $x \in (0, 1)$ with $f(x) = 0, \cos x = x$. In fact, since f is easily seen to be strictly decreasing in this interval, there is a *unique* such x. We could even approximate x to arbitrary accuracy by using, say, Newton's Method from calculus. Note that you learn nothing from applying the Intermediate Value Theorem to either $\cos x$ or x in this example; the key is to apply it to the difference $\cos x - x$.

As a “real-life” application, imagine that mountaineer hikes up a mountain, reaching the top in 24 hours; he then walks down the mountain, following the same path in reverse, the following day. No matter how irregularly he hikes (for example, how many breaks he takes and how long he rests at each break), his positions $f(t), g(t)$ along the trail at a fixed time t for each of the days are continuous functions of t. If we attach coordinates to these positions so that the bottom of the mountain is labelled 0, the top 1, then $f(0) - g(0)$ is negative while $f(24) - g(24)$ is positive. Hence there is a time t_0 of day such that the hiker is at the same spot on the trail on both days at time t_0.

Now how would you explain this last result to your grandmother (or to someone who knows nothing about mathematics)? You could invoke the following thought experiment: imagine two hikers hiking on the same day, say Monday, one following the path of our hiker going up on Monday, the other tracing the path he will follow on Tuesday, but doing this on Monday. The two hikers must meet along the trail. The time t_0 that they meet then has the required property.

Turning now (as I did in the last lecture) to discontinuous functions, given a function f defined but discontinuous at a point a, I introduce a measure of the discontinuity of f at a. For all $\delta > 0$, let M_δ be the supremum of all differences $f(y) - f(z)$ as y, z ranges over the interval $(a - \delta, a + \delta)$ intersected with the domain of f; take $M_\delta = \infty$ if the differences are not bounded above on $(a - \delta, a + \delta)$. Then $M_{1/n}$ is a nonnegative decreasing function of n (since for all n one of the differences is 0). Let $\omega_a f$ be the limit of $M_{1/n}$ as $n \to \infty$ (or set $\omega_a f = \infty$ if $M_{1/n} = \infty$ for all n). We call $\omega_a f$ the oscillation of f at a.

Then we have $\omega_a f = 0$ if and only if f is continuous at a. Indeed, if f is discontinuous at a, then for some $\epsilon > 0$ and all positive integers n, there is a_n with $|a_n - a| < 1/n$ and $|f(a_n) - f(a)| > \epsilon$, whence it follows at once that $\omega_a f \geq \epsilon$. Conversely, if $\omega_a f > \epsilon$, then for all n there must be a_n, b_n with $|a_n - a| < 1/n, |b_n - a| < 1/n$, and $f(a_n) - f(b_n) > \epsilon$, whence one of $|f(a_n) - f(a)|, |f(b_n) - f(a)|$ is greater than $\epsilon/2$ and f is discontinuous at a. Moreover, it is not difficult to show that for any f and any integer n, the set of points a with $\omega_a f > 1/n$ is closed in $\mathbb{R}$ (see Exercise 4.6.8, p. 143). The upshot is that for any f, the set of points in its domain D at which it is discontinuous is a countable union of closed subsets of D. Using this fact, one can show (as previously mentioned) that there is no real-valued function continuous at every rational number but discontinuous at every irrational one.