# Lecture 4-28: Limits of functions and continuity

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I now finally take up functions, the bread and butter of calculus. The first order of business is to decide what the limit of a function should be at a point, generalizing the limit of a sequence.

## Definition 4.2.1, p. 116

Given a real-valued function $f$ defined on a subset $A$ of $\mathbb{R}$ and a point $x$ of $A$, we say that $f$ *has the limit $L$ as $x$ approaches $a$* and write $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$ and $x \in A$.

This definition is similar to that of a limit of a sequence, but note that the real parameter $\delta$ replaces the index parameter $N$ in that definition. Also note that the hypothesis of the requirement on $\delta$ reads $0 < |x - a| < \delta$, not $|x - a| < \delta$; thus the value of $f$ at $a$, even if undefined, is entirely irrelevant to the existence of the limit. An equivalent way to formulate this hypothesis is to say that whenever $(s_n)$ is a sequence of points in $A$ different from $x$ that converges to it, then $f(s_n)$ converges to $L$.

Thus for example we have $\lim_{x\to 2}(x^2+3)=7$, since given $\epsilon > 0$ we let $\delta = \min(\epsilon/5, 1)$. Then if $|x-2| < \delta$ we have in particular that $|x-2| < 1, |x+2| < 5$, whence $|x^2 = 3 - 7| = |x^2 - 4| = |x-2||x+2| < 5\epsilon/5 < \epsilon$, as desired. Note that in this case there was no need to assume that $0 < |x-2| < \delta$, since $|x^2+3-7| < \epsilon$ even if $x = 2$.

On the other hand, $\lim_{x\to 0} \sin 1/x$ does not exist; that is, there is no $L$ such that $\lim_{x\to 0}\sin 1/x = L$. To prove this, suppose contrarily that such an $L$ exists. Taking $\epsilon = 1/2$, we note for any $\delta > 0$ that $0 < \frac{2}{(4n+1)\pi}, \frac{2}{(4n+3)\pi} < \delta$ for sufficiently large $n$. But the values of $\sin 1/x$ at $x = \frac{2}{(4n+1)\pi}, \frac{2}{(4n+3)\pi}$ are respectively 1 and $-1$ for any $n$. We would therefore have to have $|1-L|, |-1-L| < 1/2$, which we already showed is impossible when we showed that $\lim_{n\to\infty}(-1)^n$ does not exist. See Example 4.2.6 on p. 119, particularly the graph given there.

One has the same limit laws for functions (with the same proofs) as for sequences, so that if $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) + g(x)) = L + M, \lim_{x\to a}(f(x) - g(x)) = L - M, \lim_{x\to a} f(x)g(x) = LM$, and $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided that $M \neq 0$ (Corollary 4.2.4, p. 119). The proof is the same as for sequences. In particular, for any constant $c$, we have $\lim_{x\to a} cf(x) = cL$ in this situation.

In evaluating a limit $\lim_{x \to a} f(x)$, one is always tempted to just plug in $a$ for $x$, so that the limit would be $f(a)$. This does not hold for arbitrary functions $f$ and numbers $a$; in intuitive terms, it may not be possible to draw the graph of $f(x)$ near the point $x = a$ without lifting the pencil from the page. There are however many functions $f$ and values $a$ for which this holds, enough that it is worth making the following definition.

### Definition 4.3.1, p. 122

We say that $f$ is *continuous at* $x = a$ if $\lim_{x \to a} f(x) = f(a)$, or equivalently for every $\epsilon > 0$ there is $\delta > 0$ with $|f(x) - f(a)| < \epsilon$ whenever $x \in A$ and $|x - a| < \delta$. We say that $f$ is *continuous (on* $A$*)* if it is continuous at every point of $A$.

One can reformulate this definition in terms of sequences, as follows.

**Theorem 4.3.2 (iii), p. 123**

A function $f$ defined at $a \in A$ is continuous at that point if and only if for all sequences $s_n$ of points in $A$ converging to $a$ we have that $f(s_n)$ converges to $f(a)$.

Indeed, if $f$ is continuous at $a$ and $s_n \to a$ as $n \to \infty$, then given $\epsilon > 0$ there is $\delta > 0$ such that $|x - a| < \delta$ and $x \in A$ imply that $|f(x) - f(a)| < \epsilon$; in turn, there is an index $N$ such that for any $n > N$ we have $|s_n - a| < \delta$, whence $f(s_n) \to f(a)$ as $n \to \infty$. Conversely, if the sequence criterion holds and $\epsilon > 0$, suppose for a contradiction that the choice $\delta = 1/n$ never satisfies the definition of continuity for any $n$, so that there is $s_n \in A, |s_n - a) < 1/n$, but $|f(s_n) - f(a)| > \epsilon$. Then we have $s_n \to a$ but $f(s_n) \not\to f(a)$, contradicting the sequence criterion.

In particular, if $a \in A$ is an isolated point, meaning that for some $\epsilon > 0$ we have $|x - a| > \epsilon$ for all $x \in A$ with $x \neq a$, then the only sequences $s_n$ of points in $A$ converging to $a$ have $s_n = a$ for all sufficiently large $n$, whence trivially $f(s_n) \to f(a)$. Thus any function on $A$ is continuous at any isolated point of $A$. The definition of continuity at a point of $A$ has substance only for limit (that is, non-isolated) points of $A$. Notice also that there are two ways that a function $f$ defined at $a \in A$ might be discontinuous there. If $\lim_{x \to a} f(x)$ exists and equals $L$, but $f(a) \neq L$, then $f$ is discontinuous at $a$ but becomes continuous there if we redefine $f(a)$ to be $L$. We say that the discontinuity of $f$ at $a$ is removable in this case. If $\lim_{x \to a} f(x)$ does not exist then no matter how $f(a)$ is defined, $f$ is not continuous at that point.

There is a chain rule for continuous functions. It states

**Theorem 4.3.9, p. 126**

If $f, g$ are functions such that $f$ is continuous at $a$ and $g$ is continuous at $f(a)$, then the composite function $g(f)$ is continuous at $a$.

Given $\epsilon > 0$ there is $\delta > 0$ such that whenever $|b - f(a)| < \delta$ and $g(b)$ is defined we have $|g(b) - g(f(a))| < \epsilon$. In turn, there is $\delta_1 > 0$ such that whenever $|x_a| < \delta_1$ and $f(a)$ is defined we have $|f(x) - f(a)| < \delta$. Putting these together, we find that whenever $|x - a| < \delta_1$ and $f(x), g(f(x))$ are both defined we have $|g(f(x)) - g(f(a))| < \epsilon$, as desired.

There is a very elegant way to formulate the definition of continuity on a set with using $\epsilon$s or $\delta$s at all. Recall for any subset $A$ of $\mathbb{R}$ we say that a subset $V$ of $A$ is open in $A$ if it is the intersection $U \cap A$ of an open subset $U$ of $\mathbb{R}$ and $A$; we define a subset of $A$ to be closed in $A$ similarly. Recall also that for any real-valued function $f$ and subset $S$ of $\mathbb{R}$ we define $f^{-1}(S)$ (the inverse image of $S$) to be the set of points $x$ such that $f(x)$ (is defined and) lies in $S$. Then we have

**Theorem**

A function $f$ is continuous (on its domain) if and only if the inverse image $f^{-1}(U)$ is open for any open subset $U$ of $\mathbb{R}$.

## Proof.

If $F$ is continuous on its domain $A$, then let $U \subset \mathbb{R}$ be open and $x \in f^{-1}(U)$, so that $f(x) = y \in U$. Choose $\epsilon > 0$ so that the open interval $(y - \epsilon, y_{+}\epsilon) \subset U$ and choose $\delta > 0$ so that the image $f((x - \delta, x + \delta) \cap A)$ of the intersection of the open interval $(x - \delta, x + \delta)$ and $A$ lies in $(y - \epsilon, y + \epsilon)$. Then $(x - \delta, x + \delta) \cap A \subset f^{-1}(U)$, so that $f^{-1}(U)$ is open in $A$, as desired. Conversely, if the openness condition holds, $x \in A$, and $\epsilon > 0$, then set $y = f(x)$. The inverse image $f^{-1}(y - \epsilon, y + \epsilon)$ is then open in $A$, whence there is $\delta > 0$ such that $(x - \delta, x + \delta) \cap A \subset f^{-1}(U)$. This says exactly that $f$ is continuous at $x$, as desired. □

In particular, using this criterion, it is very easy to show that any composition of continuous functions is continuous (Theorem 4.3.9, p. 126).

I will explore the consequences of this criterion in conjunction with connected and compact subsets of $\mathbb{R}$ in future lectures. For now, I shift the focus from nice (that is, continuous) functions to nasty ones.

## Example

Let $g(x)$ be the function defined via $g(x) = 1$ if $x \in \mathbb{Q}$, $g(x) = 0$ if $x \notin \mathbb{Q}$. (See p. 112 of the text.). Then I claim that $f$ is discontinuous (not continuous) at every $x \in \mathbb{R}$. To prove this, observe first that there is a sequence $(s_n)$ of rational numbers converging to $x$, since $\mathbb{Q}$ is dense in $\mathbb{R}$. Similarly, there is a sequence $(t_n)$ of rational numbers converging to $x - \sqrt{2}$, and then $(t_n + \sqrt{2})$ is a sequence of irrational numbers converging to $x$. We have $g(x_n) = 1$ for all $n$ while $g(t_n + \sqrt{2}) = 0$ for all $n$, whence in any event either $g(s_n)$ fails to converge to $g(x)$ or $g(t_n + \sqrt{2})$ fails to converge to $g(x)$. Thus $g$ is not continuous at $x$. The graph of $g$ is visually indistinguishable from the union of the two lines $y = 0$ and $y = 1$ in the $xy$-plane; certainly it cannot be drawn without lifting the pencil from the paper.

## Example

A considerably more subtle example is the following one, due to Thomae (see p. 114 of the text). Define $t(x)$ via $t(0) = 1, t(x) = 1/n$ if $x = m/n \in \mathbb{Q}$ in lowest terms with $n > 0, x \neq 0$, and finally $t(x) = 0$ if $x \notin \mathbb{Q}$. Then I claim that $t$ is continuous at all irrational $x$, but discontinuous at all rational $x$. To prove this, suppose first that $x$ is rational. Then $t(x) \neq 0$, by the definition of $t$, but there is a sequence $s_n$ of irrational numbers converging to $x$ with $g(s_n) = 0$ for all $n$, so that $t$ is discontinuous at $x$. On the other hand if $x \notin \mathbb{Q}$, then given $\epsilon > 0$ there is $N \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$ for $n > N$. For $1 \leq i \leq N$ there is a minimum distance $d_i > 0$ between $x$ and any multiple of $1/i$. Taking $\delta$ to be the minimum of $d_1, \ldots, d_N$, we find that $|t(y)| < \epsilon$ if $|y - x| < \delta$, whether or not $y$ is rational. Hence $t$ is continuous at $x$, as claimed. It turns out, by the way, that the opposite behavior is impossible: there is no function $f$ continuous at every $x \in \mathbb{Q}$ but discontinuous at every $x \notin \mathbb{Q}$.

I conclude by observing that it is *false* that the *image* $f(U)$ of an open set $U$ under a continuous function $f$ is open (consider for example constant functions). Likewise the image $f(C)$ of a closed set $C$ under a continuous function $f$ need not be closed (take $f(x) = 1/x$ and $C$ to be the set of positive integers),