Lecture 4-23: Review

April 23 2025

This lecture will be entirely devoted to review for the midterm on Friday. I begin with the fundamental property that distinguished the real numbers from the rational ones and makes it possible to do calculus on the former, namely the Least Upper Bound Property that every nonempty set S of real numbers that is bounded above has a least upper bound (or supremum); likewise every nonempty set S of real numbers that is bounded below has a greatest lower bound (or infimum).

I constructed the real number r as a so-called cut C_r of rational numbers, more precisely taking C_r to consist of the rational numbers strictly less than r. Then $r \leq s$ if and only if $C_r \subseteq C_s$. The cut C corresponding to any nonempty set S of real numbers that is bounded above is simply the union of the cuts C_s corresponding to each element s of S, so that the Least Upper Bound Property is satisfied. Standard arithmetic operations on real numbers can be performed on cuts, removing largest elements as necessary; for example, the negative of a cut C consists by definition of the negatives of all rational numbers *not* in C, with the largest number of this last set removed if there is one.

Having constructed the real numbers, I turned to sequences and series. A sequence $s = s_n$ is just a choice of real numbers s_n, one for every $n \in \mathbb{N}$. A series $\sum_{i=1}^{\infty} t_i$ is just by definition the sequence s_n of its partial sums, where $s_n = \sum_{i=1}^{n} t_i$. The sequence s_n converges to the (finite) limit L if for every $\epsilon > 0$ there is an index N such that $|s_n - L| < \epsilon$ whenever $n \geq N$; a series $\sum_{i=1}^{\infty} t_i$ thus converges to its finite sum S if and only if for every $\epsilon > 0$ there is an index N such that $|\sum_{i=1}^{n} t_i - S| < \epsilon$ whenever $n \geq N$. Be careful not to confuse a sequence (t_i) with the series $\sum_{i=1}^{\infty} t_i$ whose terms are the t_i. For example, if $t_i = 1/i$, then $t_i \to 0$ as $i \to \infty$, but the series $\sum_{i=1}^{\infty} t_i$ diverges to ∞; that is, its partial sums get arbitrarily large.

The Monotone Convergence Theorem guarantees that a monotone sequence (t_n), that is, one such that either $t_n \leq t_{n+1}$ for all n or $t_n \geq t_{n+1}$ for all n, converges if and only if it is bounded, so that there is $M \in \mathbb{R}$ with $|t_n| < M$ for all n. As an immediate consequence, a series $\sum_{i=1}^{\infty} a_i$ with $a_i \geq 0$ for all i converges if and only if its partial sums are bounded. A direct calculation shows that the geometric series $\sum_{i=0}^{\infty} r^i$ converges if and only if $|r| < 1$; its sum in this case is $\frac{1}{1-r}$. On the other hand, the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ has unbounded partial sums and accordingly diverges.

Any sequence a_n always has a monotone subsequence $b_k = a_{n_k}$ (so that the indices n_i satisfy $n_1 < n_2 < \ldots$). In particular, if a_k is bounded, so that the a_k all lie in a closed bounded interval $[a, b]$, then a_k has a subsequence converging to some $c \in [a, b]$; this is the Bolzano-Weierstrass Theorem. More generally, a sequence s_n converges if and only if it is Cauchy in the sense that for every $\epsilon > 0$ there is an index N with $|s_n - s_m| < \epsilon$ for $n, m > N$. One can give an alternative construction of the real numbers using Cauchy sequences rather than (Dedekind) cuts.

Turning now to infinite series, it follows at once from the above results that a series $\sum a_k$ with $a_k \geq 0$ for all but finitely many k either has $s_n \to \infty$ as $n \to \infty$, where $s_n = \sum_{k=1}^{n} a_k$, or (s_n) converges; the latter holds if and only if the set of s_n is bounded. We have two main tests to verify this last hypothesis: the Comparison Test, which says that if $\sum a_k, \sum b_k$ are two series with $0 \leq a_k \leq b_k$ for all but finitely many k and if $\sum b_k$ converges, so does $\sum a_k$; if $\sum a_k$ diverges, so does $\sum b_k$. This result is considerably broadened by the Limit Comparison Test, which says that if the series $\sum a_k, \sum b_k$ have $a_k, b_k \geq 0$ for all but finitely many k, and if $L = \lim_{k\to\infty} \frac{a_k}{b_k}$ (exists and) is finite and nonzero, then $\sum a_k$ converges if and only if $\sum b_k$ does.

To get started with.applying these tests, we need a library of series with known convergence behavior. The p-series $H_p = \sum_{k=1}^{\infty} \frac{1}{k^p}$ (for constant p) provide such a library: using the Cauchy Condensation Test, which says that a series $\sum a_k$ with $a_k \geq a_{k+1} \geq \dots$ and $a_k \to 0$ as $k \to \infty$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_k$ converges, together with a suitable geometric series, we get that the series H_p converges if and only if $p > 1$.

Recall also the standard limit laws for sequences: if $s_n \to s, t_n \to t$ as $n \to \infty$, then $s_n + t_n \to s + t, s_n - t_n \to s - t, s_n t_n \to st$, as $n \to \infty$. Also $\frac{s_n}{t_n} \to \frac{s}{t}$ as $n \to \infty$, provided that $t \neq 0$. For series, if $\sum a_n, \sum b_n$ converge to S, T, respectively, then $\sum(a_n + b_n), \sum(a_n - b_n)$ converge to $S + T, S - T$, respectively. There is however no relationship between the convergence of $\sum a_n, \sum b_n$ and that of $\sum a_n b_n$.

For general series $\sum a_n$ (not necessarily with nonnegative terms a_n), we have that $\sum a_n$ converges whenever it converges absolutely, so that $\sum |a_n|$ also converges. If $\sum a_n$ converges but not absolutely, then we say it converges conditionally. We have the Alternating Series Test, which says that $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ converges whenever $a_1 \geq a_2 \geq \ldots, a_k \to 0$ as $k \to \infty$. Much more generally, we have the Dirichlet Test, which says that $\sum x_k y_k$ converges if the partial sums of $\sum x_k$ are bounded and in addition $y_1 \geq y_2 \geq \ldots, y_k \to 0$ as $k \to \infty$.

I ended up with some basic facts about the topology of the real line; that is, about open and closed subsets of real numbers. A subset U of $\mathbb{R}$ is open if whenever $x \in U$ there is $\epsilon > 0$ (depending on x) with the open interval $(x - \epsilon, x + \epsilon) \subset U$; a subset C of $\mathbb{R}$ is closed if every convergent sequence of points x_n of C has its limit x also in C. Any union of open sets is open, as is any *finite* intersection of open sets. Correspondingly, any intersection of closed sets is closed, as is any finite union of closed sets.

A subset S of $\mathbb{R}$ is called connected if one *cannot* write S as the union of its nonempty intersections with U and C, where U and C are disjoint subset of $\mathbb{R}$ with U open and C closed. Using the Least Upper Bound property, it is not difficult to show that any interval $I \subset \mathbb{R}$, open, half-open, or closed, and bounded or unbounded, is connected. Conversely, a subset of $\mathbb{R}$ is connected if and only if it is such an interval.

A subset S of $\mathbb{R}$ is compact if it is both closed and bounded, or equivalently if and only if every sequence of points in S has a convergent subsequence whose limit lies in S. The notions of connectedness and compactness will both play important roles later on in the course when I discuss continuous functions.

Finally, some logistics: you will do all your work on the test paper and are permitted one sheet (front and back) of handwritten notes.